Haar measure on the unitary groups

The aim of this text is not to provide an introduction to group theory. The explanation of the general concept of the group-invariant (Haar) measure can be found for instance in the book *Theory of group representations and applications* by Barut and Raczka. Instead, my goal is to provide some details of the derivation of the Haar measure on the unitary groups $U(N)$ and $SU(N)$, which are often used in quantum field theory applications. The derivation proceeds in two steps. First I will find an invariant metric on the group. Then I will use this metric to construct the invariant measure, and therefrom derive the distribution of eigenvalues. Finally, I will show that this distribution in the case of $SU(3)$ can be rewritten in terms of the trace of the group element in the fundamental representation, that is, the character of this representation.

1. Invariant metric. The vector space of all complex matrices possesses a natural scalar product, defined by $\langle A|B \rangle = \text{Tr}(A^\dagger B)$. This in turn gives rise to the norm, $\|A\|^2 = \langle A|A \rangle$, and the distance of two matrices, $s(A, B) = \|A - B\|$. This distance is obviously invariant under both left and right multiplication by a unitary matrix. It therefore defines an invariant metric, when restricted to the group of unitary matrices. To proceed further, we calculate the infinitesimal squared distance, whose coefficients in a conveniently chosen coordinate system determine the metric tensor.

To that end, note that every unitary matrix $U$ can be spectrally decomposed as $U = W\Lambda W^\dagger$, where $W$ itself is also unitary and $\Lambda$ is diagonal. Its entries are mere phases, $\lambda_i = e^{i\theta_i}$, due to the unitarity of $U$. Provided that all eigenvalues $\lambda_i$ are different, the matrix $W$ is unique up to a right multiplication by a unitary diagonal matrix, that commutes with $\Lambda$. It therefore encompasses $N^2 - N$ real parameters ($N$ being the dimension of $U$), which together with the $N$ phases in $\Lambda$ give the $N^2$ independent parameters of $U$. An infinitesimal shift $dU$ on the group changes these matrices by $d\Lambda$ and $dW$, the latter being constrained by the unitarity,

$$W dW^\dagger + dW W^\dagger = 0.$$  \hfill (1)

The shift of $U$ therefore becomes

$$dU = dW \Lambda W^\dagger + W d\Lambda W^\dagger + W \Lambda dW^\dagger = W W^\dagger dW \Lambda W^\dagger + W d\Lambda W^\dagger - W \Lambda W^\dagger dW W^\dagger = W (d\Lambda + [W^\dagger dW, \Lambda]) W^\dagger,$$

where we used (1) in the first step. The infinitesimal squared distance thus reads

$$ds^2 = \text{Tr}(dU^\dagger dU) = \text{Tr}(d\Lambda^\dagger d\Lambda) + \text{Tr}([W^\dagger dW, \Lambda]^\dagger [W^\dagger dW, \Lambda]) + 2 \text{Re} \text{Tr}(d\Lambda^\dagger [W^\dagger dW, \Lambda]).$$

The last term is easily seen to vanish since $\text{Tr}(d\Lambda^\dagger [W^\dagger dW, \Lambda]) = \text{Tr}([W^\dagger dW, \Lambda] d\Lambda^\dagger)$, and $[\Lambda, d\Lambda^\dagger] = 0$ thanks to the fact that $\Lambda$ is diagonal. Denoting $W^\dagger dW$ as $d\Omega$, remembering that $d\Omega$ is antihermitian in view of (1), and finally observing that

$$[d\Omega, \Lambda]_{ij} = (\lambda_j - \lambda_i) d\Omega_{ij},$$
we conclude that
\[
\text{ds}^2 = \sum_i d\theta_i^2 + 2 \sum_{i>j} |\lambda_i - \lambda_j|^2 |d\Omega_{ij}|^2.
\] (2)

2. Invariant measure and distribution of eigenvalues. From the invariant metric, \( \text{ds}^2 = g_{\alpha\beta}(x) dx^\alpha dx^\beta \), in a given set of coordinates, one may construct the invariant measure as
\[
d\mu(x) = \sqrt{\det g(x)} \prod_\alpha dx^\alpha.
\]

One would be therefore tempted to identify \(d\theta_i\) and \(d\Omega_{ij}\) with the coordinate differentials and simply multiply the prefactors in (2) to get the determinant of the metric. Even though this eventually does lead to the correct result, one had better be a bit more careful. The reason is that the differentials \(d\Omega_{ij}\) are in general not exact, and thus do not give rise to well-defined coordinates “\(\Omega_{ij}\)”. Instead, let us parameterize the matrix \(W\) as
\[
W = \exp(it_\alpha T_\alpha),
\]
where \(T_\alpha\) is the set of \(N^2\) generators of \(\mathbb{U}(N)\). As explained above, using the freedom in the definition of \(W\), the parameters \(t_\alpha\) can be expressed in terms of \(N^2 - N\) independent coordinates, \(w_a\). In fact, in the neighborhood of unity we may take \(t_a = w_a\) for \(a = 1, \ldots, N^2 - N\) (corresponding to the offdiagonal generators), and \(t_\alpha = 0\) for \(\alpha = N^2 - N + 1, \ldots, N^2\) (corresponding to the diagonal generators). However, we do not need to know the precise dependence \(t_\alpha(w)\). Using now the Baker–Campbell–Hausdorff formula leads to the expression
\[
d\Omega_{ij} = (W^\dagger dW)_{ij} = i(T_\alpha)_{ij} f_{aa}(w) dw_a,
\]
where \(f_{aa}(w)\) are some functions of the “angles” \(w_a\), which we do not need to specify. There are \(N^2 - N\) independent differentials, \(dw_a\), or Re \(d\Omega_{ij}\) and Im \(d\Omega_{ij}\) with \(i > j\). These two sets are obviously related by the linear transformation with real coefficients, \(d\Omega_a = Q_{ab}(w) dw_a\), where \(d\Omega_a\) stands collectively for the real and imaginary parts of \(d\Omega_{ij}\).

The crucial observation is that the determinant of the metric in the coordinates \(\theta_i, w_a\) still factorizes into \(\theta\)- and \(w\)-dependent parts. Concretely, we find
\[
det g \sim (\det Q)^2 \prod_{i>j} |\lambda_i - \lambda_j|^4,
\]
up to an irrelevant numerical prefactor. The fourth power in the last term comes from the fact that the real and imaginary parts of \(d\Omega_{ij}\) are independent differentials. In many physical applications we need to integrate just functions which are invariant under the adjoint action of the group on itself, hence depend on the eigenvalues \(\lambda_i\) only. We can then trivially integrate out the angles \(w_a\) and obtain the final result: the distribution of the eigenvalues on the unitary group is given by the invariant measure
\[
d\mu(\theta) = \prod_{i>j} |e^{i\theta_i} - e^{i\theta_j}|^2 \prod_i d\theta_i.
\]
(3)
This formula is strictly speaking valid for $U(N)$. However, it is obvious from the above derivation that the requirement of unit determinant in $SU(N)$ is equivalent to imposing the constraint $\sum_i \theta_i = 0$ mod $2\pi$, which is easily implemented using a $\delta$-function. The density distribution of the eigenvalues on $SU(N)$ is therefore given by the same formula (3), keeping in mind that the independent variables are now only $\theta_1, \ldots, \theta_{N-1}$.

3. Expression in terms of the character of the fundamental representation. Recall from Eq. (3) that the distribution of eigenvalues on $SU(N)$ is given by the invariant measure function

$$H(\theta) = \prod_{i>j} |e^{i\theta_i} - e^{i\theta_j}|^2,$$

augmented by the constraint $\sum_i \theta_i = 0$ mod $2\pi$. The product of differences of the eigenvalues can be expressed as a Vandermonde determinant: for any set of complex numbers $z_i$, we have

$$\prod_{i>j} (z_i - z_j) = \det \mathcal{M} \equiv \begin{vmatrix} 1 & z_1 & \cdots & z_1^{N-1} \\ 1 & z_2 & \cdots & z_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_N & \cdots & z_N^{N-1} \end{vmatrix}.$$

Writing the absolute value squared of the determinant as $|\det \mathcal{M}|^2 = \det \mathcal{N}$, where $\mathcal{N}_{ij} = (\mathcal{M}^\dagger \mathcal{M})_{ij} = \sum_k z_i^{-1} z_j^{-1}$. In our case $z_k = e^{i\theta_k}$, hence $|z_k| = 1$ and $\mathcal{N}_{ij} = \sum_k z_j^{-i}$. Concretely for $N = 3$ the result becomes

$$H(\theta) = \left| \sum_{k} z_k \right|^2 = \left| \sum_{k} \bar{z}_k \right|^2 = \left| \sum_{k} z_k \right|^2 - 2 \left| \sum_{k} z_k \bar{z}_k \right|^2 = 9\ell^2 - 6\bar{\ell}.$$

(We used the fact that $z_1 z_2 z_3 = 1$.) This in turn easily leads to the final result,

$$H(\ell, \bar{\ell}) = 27 \left| \begin{array}{ccc} 1 & \ell & 3\ell^2 - 2\bar{\ell} \\ \ell & 1 & \ell \\ 3\ell^2 - 2\ell & \ell & 1 \end{array} \right| = 27 \left[ 1 - 6\ell\bar{\ell} + 4(\ell^3 + \bar{\ell}^3) - 3(\ell \bar{\ell})^2 \right].$$

Therefore, for $SU(3)$ the Haar measure can be expressed solely in terms of the character of the fundamental representation, $\ell$, and its complex conjugate. This is in general not possible for $N > 3$. 