Algebra of block matrices

(i) Inverse of a triangular block matrix:

\[
\begin{pmatrix}
A & B \\
0 & D
\end{pmatrix}^{-1} = \begin{pmatrix}
A^{-1} & -A^{-1}BD^{-1} \\
0 & D^{-1}
\end{pmatrix}, \quad \begin{pmatrix}
A & 0 \\
C & D
\end{pmatrix}^{-1} = \begin{pmatrix}
A^{-1} & 0 \\
-D^{-1}CA^{-1} & D^{-1}
\end{pmatrix}.
\]

(ii) Triangular decomposition:

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
A - BD^{-1}C & B \\
0 & D
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
D^{-1}C & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
A & 0 \\
C & D - CA^{-1}B
\end{pmatrix} \begin{pmatrix}
1 & A^{-1}B \\
0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 \\
CA^{-1} & 1
\end{pmatrix} \begin{pmatrix}
A & B \\
0 & D - CA^{-1}B
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & BD^{-1} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
A - BD^{-1}C & 0 \\
0 & C
\end{pmatrix}.
\]

(iii) Combining these decompositions with the formulas (i) for the triangular matrices, one obtains several equivalent expressions for the inverse of a general block matrix:

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^{-1} = \begin{pmatrix}
K & L \\
M & N
\end{pmatrix},
\]

where

\[
K = (A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1},
\]

\[
L = -(A - BD^{-1}C)^{-1}BD^{-1} = -A^{-1}B(D - CA^{-1}B)^{-1},
\]

\[
M = -D^{-1}C(A - BD^{-1}C)^{-1} = -(D - CA^{-1}B)^{-1}CA^{-1},
\]

\[
N = (D - CA^{-1}B)^{-1} = D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} + D^{-1}.
\]

(iv) Reduction of the determinant of a block matrix:

\[
\det \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \det (DA - DBD^{-1}C) = \det (AD - ACA^{-1}B).
\]
Algebra of energy projectors

Basic properties of the projectors on the subspaces of the Dirac Hamiltonian, corresponding to positive and negative energy eigenstates.

(i) Definition and orthogonality:

$$\Lambda^e_k = \frac{1}{2} \left[ 1 + \frac{e}{\epsilon_k} \gamma_0 (\gamma \cdot k + m) \right], \quad \text{where} \quad \epsilon_k = \sqrt{k^2 + m^2}, \quad e, f = \pm,$$

$$\Lambda^e_k \Lambda^f_k = \delta_{ef}, \quad \text{Tr}(\Lambda^e_k \Lambda^f_q) = 1 + ef \frac{m^2 + k \cdot q}{\epsilon_k \epsilon_q}.$$

(ii) Selected properties:

$$\mathcal{H}_k \equiv \alpha \cdot k + \gamma_0 m = \sum_{e=\pm} e \epsilon_k \Lambda^e_k,$$

$$\mathcal{k} - m \pm \mu \gamma_0 = \gamma_0 \sum_{e=\pm} (k_0 \pm \mu - e \epsilon_k) \Lambda^e_k,$$

$$\gamma_0 \Lambda^e_k \gamma_0 = \Lambda^e_{-k}, \quad \gamma_5 \gamma_0 \Lambda^e_k \gamma_0 \gamma_5 = \Lambda^{-e}_k.$$

Nambu–Gorkov formalism: relativistic case I

(i) Definition of the Nambu doublet:

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \psi^T(x) \end{pmatrix}.$$

(ii) Reflection symmetry of the fermion propagator:

$$S = \begin{pmatrix} G^+ & \Xi^- \\ \Xi^+ & G^- \end{pmatrix}, \quad \text{where}$$

$$\Xi^+(p) = \gamma_0 [\Xi^-(p)]^T \gamma_0, \quad G^-(p) = -[G^+(p)]^T.$$

(iii) Inverse of the matrix propagator: $S = (S^{-1}_0 + \Sigma)^{-1}$, where

$$S_0 = \begin{pmatrix} G^+_0 & 0 \\ 0 & G^-_0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma^+ & \Phi^- \\ \Phi^+ & \Sigma^- \end{pmatrix},$$

$$G^\pm = \left\{ (G^\pm_0)^{-1} + \Sigma^\pm - \Phi^\pm \left[ (G^\mp_0)^{-1} + \Sigma^\mp \right]^{-1} \Phi^\pm \right\}^{-1},$$

$$\Xi^\pm = - \left[ (G^\mp_0)^{-1} + \Sigma^\mp \right]^{-1} \Phi^\pm G^\pm.$$
Nambu–Gorkov formalism: relativistic case II

(i) Definition of the Nambu doublet:
\[ \Psi(x) = \begin{pmatrix} \psi(x) \\ \psi^C(x) \end{pmatrix}. \]

(ii) Reflection symmetry of the fermion propagator:
\[ S = \begin{pmatrix} G^+ & \Xi^- \\ \Xi^+ & G^- \end{pmatrix}, \quad \text{where} \]
\[ \Xi^+(p) = \gamma_0[\Xi^-(p)]^\dagger \gamma_0, \quad G^-(p) = -C[G^+(p)]^T C^{-1}. \]

Nambu–Gorkov formalism: nonrelativistic case

(i) Definition of the Nambu doublet:
\[ \Psi(x) = \begin{pmatrix} \psi(x) \\ \psi^*(x) \end{pmatrix}. \]

(ii) Reflection symmetry of the fermion propagator:
\[ S = \begin{pmatrix} G^+ & \Xi^- \\ \Xi^+ & G^- \end{pmatrix}, \quad \text{where} \]
\[ \Xi^+(p) = [\Xi^-(p)]^\dagger, \quad G^-(p) = -[G^+(p)]^T. \]

Grassmann integration on the Nambu space

Working with fermions in the Nambu formalism, one often encounters a (functional) integral of the following general form,
\[ \int dz \, d\bar{z} \exp \left( \frac{i}{2} \Psi M \bar{\Psi} \right), \quad (1) \]
\[ \text{where} \quad \Psi = \begin{pmatrix} \bar{z} \\ \bar{\bar{z}} \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \]
The Grassmann spinors \( z, \bar{z} \) may carry further indices so that \( A, B, C, D \) themselves are in general matrices. Let us recall the formula for the Gaussian Grassmann integral,

\[
\int d\z d\bar{\z} \exp(zA\bar{z}) = \det A,
\]

up to a possible minus sign depending on the exact definition of the multivariate Grassmann measure. In view of this expression, one is tempted to conclude that the integral (1) equals \( \sqrt{\det M} \), since the size of the matrix \( M \) is double the size of \( A \) in (2). In other words, the integration measure in (1) can be written as \( d\z d\bar{\z} = d\Psi \) so that one actually has only half of the integration variables needed for the Gaussian integral. In the following we will show that indeed, the integral (1) is equal to \( \sqrt{\det M} \), under certain assumptions on the matrix \( M \).

To understand what kind of restrictions may apply to \( M \), let us for a while assume that there is no mixing, that is, \( B = C = 0 \). Now the matrix \( D \) is essentially equivalent to \( A \) because \( \bar{z}Dz = z(-D^T)\bar{z} \). The integral (1) therefore reduces to a Gaussian with the matrix \( \frac{1}{2}(A - D^T) \).

Note that there is a freedom in the definition of \( A \) and \( D \), since shifting \( A \to A + \Lambda \) and \( D \to D + \Lambda^T \) with an arbitrary matrix \( \Lambda \) does not affect the integrand in (1). However, at the same time we have \( \sqrt{\det M} = \sqrt{\det(AD)} \), so that the claimed result is only reproduced when \( D = -A^T \). That means, the \( z_i\bar{z}_j \) part of the integrand must be split “symmetrically” between \( A \) and \( D \) in order to obtain the desired result.

Let us now proceed to the proof. First, we rewrite the integral (1) as

\[
\xi = \int dZ \exp \left( \frac{1}{2}ZNZ \right),
\]

where \( Z \) stands for \( (z, \bar{z}) \) and \( N = \begin{pmatrix} B & A \\ D & C \end{pmatrix} \). Second, we double the number of variables by writing this as

\[
\xi^2 = \int dZ dY \exp \left[ \frac{1}{2}(ZNZ + NYN) \right] = \int dZ dY \exp \left[ \frac{1}{2} \begin{pmatrix} Z & Y \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} Z \\ Y \end{pmatrix} \right].
\]

Next we perform a unitary transformation in the \((Z,Y)\) space, generated by the matrix \( U = \frac{\sigma_i}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \). This transforms the block-diagonal matrix in the exponent to \( U^T \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} U = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix} \). The desired integral thus becomes

\[
\xi^2 = \int dZ dY \exp \left[ \frac{1}{2}(ZNY + YNZ) \right] = \int dZ dY \exp \left[ \frac{1}{2}Z(N - N^T)Y \right].
\]

This is already a standard Gaussian integral of the form (2). Finally, assuming that the matrix \( N \) is antisymmetric, we recover the desired result \( \xi^2 = \det N \). This is equal to \( \det M \) (up to a minus sign), since \( M = N \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Obviously, the necessary condition for this result to hold, i.e., the antisymmetry of \( N \), is equivalent to the antisymmetry of \( B \) and \( C \), and \( D = -A^T \).