Singular Schrödinger operators and Robin billiards: geometry, spectra and asymptotic expansions

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Leaky quantum graphs and their generalizations

The first main object of interest in this talk are singular Schrödinger operators that can formally written as

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We can regard them as *waveguides* of a sort, with a finite size of the transverse localization, and *building blocks* of more complicated structures

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- Some open questions

Definition of the Hamiltonian

The easiest way to introduce a δ-interaction in the case $\operatorname{codim} \Gamma = 1$ is to employ the quadratic form,

$$\psi \mapsto \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)} - \alpha \int_\Gamma |\psi(x)|^2 \mathrm{d}x\,,$$

which is closed and below bounded in $W^{1,2}(\mathbb{R}^d)$, $d = 2, 3$; the second term makes sense in view of the standard Sobolev embedding.

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For *smooth manifolds* we can use an alternative definition by boundary conditions: $H_{\alpha,\Gamma}$ acts as $-\Delta$ on functions from $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

$$\frac{\partial\psi}{\partial n}(x)\bigg|_+ - \frac{\partial\psi}{\partial n}(x)\bigg|_- = -\alpha\psi(x)$$

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Moreover, for $\operatorname{codim}\Gamma = 1$ one can consider other, more singular interactions. The prime example is a *δ'-interaction* supported by Γ in which the roles of ψ and $\frac{\partial\psi}{\partial n}$ are switched; more about that a bit later.

The case $\operatorname{codim} \Gamma = 2$

This is more complicated but one can use again boundary conditions, appropriately modified. Furthermore, for an infinite curve Γ corresponding to a map $\gamma : \mathbb{R} \to \mathbb{R}^3$ we have to assume in addition that there is a tubular neighbourhood of Γ which *does not intersect itself*

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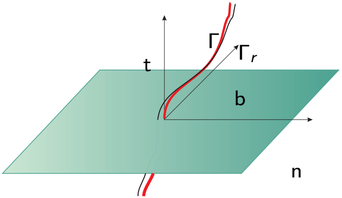
We employ *Frenet's frame* $(t(s), b(s), n(s))$ for Γ. Given $\xi, \eta \in \mathbb{R}$, we set $r = (\xi^2 + \eta^2)^{1/2}$ and define family of "shifted" curves



$$\Gamma_r \equiv \Gamma_r^{\xi\eta} := \{\, \gamma_r(s) \equiv \gamma_r^{\xi\eta}(s) := \gamma(s) + \xi b(s) + \eta n(s) \,\}$$

The case $\operatorname{codim}\Gamma = 2$, continued

The restriction of $f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma)$ to Γ_r is well defined for small r; we say that $f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma) \cap L^2(\mathbb{R}^3)$ belongs to Υ if

$$\begin{aligned} \Xi(f)(s) &:= -\lim_{r\to 0} \frac{1}{\ln r} f\restriction_{\Gamma_r}(s)\,, \\ \Omega(f)(s) &:= \lim_{r\to 0} \left[f\restriction_{\Gamma_r}(s) + \Xi(f)(s) \ln r \right]\,, \end{aligned}$$

exist a.e. in $\mathbb{R}$, are independent of the direction $\frac{1}{r}(\xi,\eta)$, and define functions from $L^2(\mathbb{R})$.

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Then the operator $H_{\alpha,\Gamma}$ has the domain

$$\{\, g \in \Upsilon :\; 2\pi\alpha\Xi(g)(s) = \Omega(g)(s) \,\}$$

and acts as

$$-H_{\alpha,\Gamma} f = -\Delta f \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \Gamma$$

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Note that absence of the interaction corresponds $\alpha = \infty$!

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On the other hand, the existence of a negative discrete spectrum is *dimension dependent*. For $d = 2$ bound states exist whenever $|\Gamma| > 0$, in particular, we have a weak-coupling expansion [Kondej-Lotoreichik'14]

$$\lambda(\alpha) = \big(C_\Gamma + o(1)\big)\, \exp\left(-\frac{4\pi}{\alpha|\Gamma|}\right) \quad \text{as} \quad \alpha|\Gamma| \to 0+$$

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On the other hand, for $d = 3$ the singular coupling must exceed a critical value. As an example, let Γ be a sphere of radius $R > 0$ in $\mathbb{R}^3$, then by [Antoine-Gesztesy-Shabani'87] we have

$$\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma}) \neq \emptyset \quad \text{iff} \quad \alpha R > 1$$

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The existence is easy, one has to consider a straight Γ and use Dirichlet bracketing. The nonexistence is a bit more involved and requires to estimate the norm of the generalized Birman-Schwinger operator.

The δ' interaction in the plane

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$$\Gamma = \big\{x_0 + r(\cos\varphi(r), \sin\varphi(r)) :\ r \in (0, R)\big\}$$

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Theorem (Jex-Lotoreichik'16)

We have $\sigma(H_{\beta,\Gamma}) \subset \mathbb{R}_+$ *if* $\beta > 2\pi r\sqrt{1+(r\varphi'(r))^2}$ *holds for all* $r \in (0,R)$.

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Question: Is $\sigma(H_{\beta,\Gamma}) \subset \mathbb{R}_+$ for *any* non-closed Γ and β large enough? According to Monique Dauge, the answer seems to be *positive*.

δ interaction supported by infinite curves

A geometrically induced spectrum may exist even if Γ is infinite and $\inf \sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) < 0$. As an example, consider a *non-straight, piecewise C^1-smooth curve* $\Gamma : \mathbb{R} \to \mathbb{R}^2$ parameterized by its arc length, assuming

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in the sector $S_\omega := \left\{(s, s') : \omega < \frac{s}{s'} < \omega^{-1}\right\}$

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Theorem (E-Ichinose'01)

Under these assumptions, $\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = [-\frac{1}{4}\alpha^2, \infty)$ and $H_{\alpha,\Gamma}$ has at least one eigenvalue below the threshold $-\frac{1}{4}\alpha^2$.

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- *Implications for graphs:* let $\tilde{\Gamma} \supset \Gamma$ in the set sense, then $H_{\alpha,\tilde{\Gamma}} \le H_{\alpha,\Gamma}$. If the essential spectrum threshold is the same for both graphs – which is often easy to establish – and Γ fits the above assumptions, we have $\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma}) \ne \emptyset$ by the minimax principle

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Consider a a C^4 smooth *curve in $\mathbb{R}^2$ without ends*, either infinite or a closed loop. In the limit $\alpha \to \infty$ the j-th eigenvalue of $H_{\alpha,\Gamma}$ behaves as

$$\lambda_j(\alpha) = -\frac{\alpha^2}{4} + \mu_j + \mathcal{O}(\alpha^{-1} \ln \alpha)$$

where μ_j is the j-th eigenvalue of

$$S_\Gamma = -\frac{\mathrm{d}}{\mathrm{d}s^2} - \frac{1}{4}\kappa(s)^2$$

on $L^2((0,|\Gamma|))$ for $\dim \Gamma = 1$, where κ is *curvature* of Γ.

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If the *attraction is strong* the motion is strongly localized transversally and the geometry of Γ is manifested. Again, we exclude self-intersections.

Consider a a C^4 smooth *curve in $\mathbb{R}^2$ without ends*, either infinite or a closed loop. In the limit $\alpha \to \infty$ the j-th eigenvalue of $H_{\alpha,\Gamma}$ behaves as

$$\lambda_j(\alpha) = -\frac{\alpha^2}{4} + \mu_j + \mathcal{O}(\alpha^{-1} \ln \alpha)$$

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Under similar hypotheses on *smoothness* and *absence of boundaries*, the claim extends to higher dimensions, specifically

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There are various extensions of these results including, in particular, asymptotic expansions for *periodic manifolds*, connected or disconnected, or *loops pierced by a magnetic flux* – cf. [Ex'08] for a review

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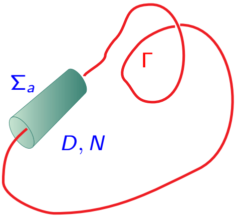
How these expansions are demonstrated: the argument has three essential ingredients. The first is *Dirichlet-Neumann bracketing* at a boundary Σ_a of a tubular neighbourhood of Γ of radius a, here sketched for a loop in $\mathbb{R}^3$



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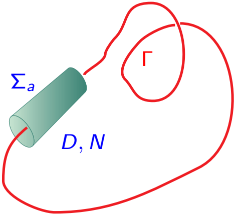

We have to care about the tube part only because the Dirichlet/Neumann Laplacian in the remaining part of $\mathbb{R}^d$ is positive

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where

$$U^{\pm}_{a} = -(1 \mp a\|\kappa\|_{\infty})^{-2}\frac{\mathrm{d}^2}{\mathrm{d}s^2} + V_{\pm}(s)$$

with PBC in the case of a loop, where $V_{-}(s) \le \frac{1}{4}\kappa^2(s) \le V_{+}(s)$ with an $\mathcal{O}(a)$ error. In other words, the $U^{\pm}_{a}$ are *$\mathcal{O}(a)$ close to S_Γ*.

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The transverse operators are associated with the forms

$$t^{+}_{a,\alpha}[f] = \int_{-a}^{a} |f'(u)|^2 \,\mathrm{d}u - \alpha |f(0)|^2$$

and $t^{-}_{a,\alpha}[f] = t^{-}_{a,\alpha}[f] - \|k\|_\infty(|f(a)|^2 + |f(-a)|^2)$ defined on $W^{1,2}_0(-a,a)$ and $W^{1,2}(-a,a)$, respectively.

Strong δ interactions, continued

Next we observe that for large α the presence of the boundaries causes just an exponentially small error:

Lemma

There is a positive c_N such that $T^{\pm}_{\alpha,a}$ has for α large enough a single negative eigenvalue $\kappa^{\pm}_{\alpha,a}$ satisfying

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In the other dimension/codimension cases the argument is analogous. If $\mathrm{codim}\,\Gamma = 2$ the transverse operator describes the Dirichlet/Neumann disc of radius r with the point interaction in the centre; the error is again exponentially small as $\alpha \to -\infty$.

Curves with ends

We have seen that the described method yields for *finite* or *semifinite* curves gives the asymptotics for the number of bound states, but fails to do that for individual eigenvalues — the difference between Dirichlet and Neumann conditions imposed on the comparison operator is too big.

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Theorem (E-Pankrashkin'14)

Suppose γ is a C^4 smooth open arc in $\mathbb{R}^2$ of length L with regular ends; then the strong-coupling limit of the j-th negative eigenvalue of $H_{\alpha,\Gamma}$ is

$$\lambda_j(\alpha) = -\frac{1}{4}\alpha^2 + \mu_j + \mathcal{O}\Big(\frac{\ln\alpha}{\alpha}\Big) \quad \text{as} \quad \alpha \to +\infty$$

where μ_j is the j-th eigenvalue of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\kappa(s)^2$ on $L^2(0,L)$ with Dirichlet b.c., where $\kappa(s)$ is as before the signed curvature of Γ at the point $s \in (0,L)$.

Curves with ends, continued

The proof starts again from bracketing estimates but no it has to be modified. The upper (Dirichlet) bound works as before, while for the lower (Neumann) we use the fact that Γ has by assumption *regular ends*, i.e. can be extended smoothly in the vicinity of the endpoints.

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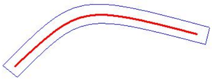
This allows us to take an *'extended' tubular neighbourhood*, at each endpoint longer by $a := \frac{6}{\alpha} \ln \alpha$. Now we loose the advantage of variable separation and the task is to show that the Neumann condition imposed at this distance from the curve end will have an effect which can be included into the error term.



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$$\psi_j(x) = \frac{1}{2\pi} \int_\Gamma K_0(\kappa_j |x - \Gamma(s)|)\, f_j(s)\, \mathrm{d}s\,,$$

where f_j is the corresponding eigenfunction of the Birman-Schwinger operator acting on $L^2(\Gamma, \mathrm{d}s)$.

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The claim of the theorem then follows from simple geometric estimates combined with the exponential decay of the Macdonald function K_0 at large distances. $\square$

Curves with ends, $\operatorname{codim} \Gamma = 2$

A similar result can be obtained for a curve arc in $\mathbb{R}^3$:

Theorem (E-Kondej'16)

Let $H_{\alpha,\Gamma}$ correspond to a finite, non-closed C^4 smooth curve in $\mathbb{R}^3$ with regular ends having length L and the global Frenet frame.

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(i) The cardinality of the discrete spectrum behaves asymptotically as

$$\sharp\sigma_{\rm disc}(H_{\alpha,\Gamma}) = \frac{L}{\pi}(-\epsilon_\alpha)^{1/2}(1+\mathcal{O}(\mathrm{e}^{\pi\alpha})) \quad \text{as} \quad \alpha \to -\infty\,.$$

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Proof is technically slightly more demanding but it follows the same basic idea as in the previous case. □

Surface with a boundary

Let $\Gamma \subset \mathbb{R}^3$ be now a C^4-smooth relatively compact orientable surface with a compact Lipschitz boundary $\partial\Gamma$. In addition, we suppose that Γ can be extended through the boundary, i.e. that there exists a larger C^4-smooth surface Γ_2 such that $\overline{\Gamma} \subset \Gamma_2$.

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As in the case of surfaces without a boundary we consider the operator $S_\Gamma = -\Delta_\Gamma^D + K - M^2$, where $-\Delta_\Gamma^D$ is Laplace-Beltrami operator on Γ, now with *Dirichlet condition* at $\partial\Gamma$, and K, M, respectively, are the corresponding *Gauss* and *mean* curvatures.

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We denote eigenvalues of this operator as μ_j^D, $j \in \mathbb{N}$, then we have

Theorem (Dittrich-E-Kühn-Pankrashkin'16)

Let Γ be as above, then for any fixed $j \in \mathbb{N}$ we have

$$\lambda_j(H_{\alpha,\Gamma}) = -\frac{\alpha^2}{4} + \mu_j^D + o(1) \quad \text{as} \quad \alpha \to \infty\,.$$

If, in addition, Γ has a C^2 boundary, then the remainder estimate can be replaced by $\mathcal{O}(\alpha^{-1} \ln \alpha)$.

Surface with a boundary, comments on the proof

As before, the upper bound is easy because one can take a layer neighbourhood of the surface Γ itself and impose the 'right', that is, Dirichlet conditions at its boundary. Using then an estimate with separated variables, we get the result.

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An alternative is to use the same idea as for the curves with ends based on Birman-Schwinger principle. This yields an error term, but since the boundary is not a more complicated object now, we have to require a C^2 smoothness in order to be able to perform the needed geometric estimates.

Planar curves supporting a δ' interaction

If $\mathrm{codim}\,\Gamma = 1$ the manifold can also support more singular interactions. One possibility is the δ'-interaction. Using the curvilinear coordinates (s, u) we can define the corresponding operator $H_{\beta,\Gamma}$ through the quadratic form

$$h_{\beta,\Gamma}[\psi] = \|\nabla\psi\|^2 - \beta^{-1}\int_\Gamma |\psi(s, 0_+) - \psi(s, 0_-)|^2\,\mathrm{d}s$$

defined on functions $\psi \in H^1(\mathbb{R}^2 \setminus \Gamma)$ as $\psi(s, u)$.

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Alternatively, one can use boundary conditions: the operator acts as Laplacian outside the interaction support,

$$(H_{\beta,\Gamma}\psi)(x) = -(\Delta\psi)(x)$$

for $x \in \mathbb{R}^2 \setminus \Gamma$, with the domain $\mathcal{D}(H_{\beta,\Gamma}) = \{\psi \in H^2(\mathbb{R}^2 \setminus \Gamma) \mid \partial_{n_\Gamma}\psi(x) = \partial_{-n_\Gamma}\psi(x) =: \psi'(x)|_\Gamma,\ -\beta\psi'(x)|_\Gamma = \psi(x)|_{\partial_+\Gamma} - \psi(x)|_{\partial_-\Gamma}\}$, where n_Γ is the outer normal to Γ and $\psi(x)|_{\partial_\pm\Gamma}$ are the appropriate traces.

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Note that the *strong-coupling limit* in this case is $\beta \to 0+$.

Strong coupling on a δ' loop

Theorem (E-Jex'13)

Let Γ be a C^4-smooth closed curve without self-intersections. Then $\sigma_{\mathrm{ess}}(H_{\beta,\Gamma}) = [0,\infty)$ and to any $n \in \mathbb{N}$ there is a $\beta_n > 0$ such that $\#\sigma_{\mathrm{disc}}(H_{\beta,\Gamma}) \ge n$ holds for $\beta \in (0,\beta_n)$. Denoting by $\lambda_j(\beta)$ the j-th eigenvalue of $H_{\beta,\Gamma}$, counted with multiplicity, we have the expansion

$$\lambda_j(\beta) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}\big(\beta|\ln\beta|\big)\,, \quad j = 1,\dots,n\,,$$

valid as $\beta \to 0_+$, where μ_j is the j-th eigenvalue of the comparison operator S, the same as before.

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valid as $\beta\to 0_+$, where μ_j is the j-th eigenvalue of the comparison operator S, the same as before. Moreover, for the counting function $\beta\mapsto \#\sigma_d(H_{\beta,\Gamma})$ we have

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A similar result holds for infinite curves, cf. [Jex'14], and for strong δ' interaction supported by surfaces *without boundary*, cf. [E-Jex'14]

A digression: Robin 'billiards'

Let Ω be an open, simply connected set in $\mathbb{R}^2$ with a closed C^4 Jordan boundary $\partial\Omega = \Gamma : [0, L] \ni s \mapsto (\Gamma_1, \Gamma_2) \in \mathbb{R}^2$, with $\gamma : [0, L] \to \mathbb{R}$ being the signed curvature of Γ. We consider the boundary-value problem

$$-\Delta f = \lambda f \;\text{ in}\, \Omega\,, \quad \frac{\partial f}{\partial n} = \beta f \;\text{ on}\, \Gamma\,,$$

with $\beta > 0$, where $\frac{\partial}{\partial n}$ is the outward normal derivative.

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A large parameter asymptotics

Since the Robin problem can be regarded as a 'one-sided version' of our singular Schrödinger operators, one can try to employ the same technique. Its naive use, however, yields only a much weaker result,

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The reason is that passing to curvilinear coordinates in the vicinity of the boundary we get in the one-sided case a boundary term containing γ. If we want estimates with separated variables we have to employ rough bounds with γ^* and γ_*. However, the lower bound can be improved by a variational technique; this yields at least the first term in the expansion:

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Theorem (E-Minakov-Parnovski'14)

In the asymptotic regime $\beta\to+\infty$ the j-th eigenvalue behaves as

$$\lambda_j(\beta)=-\beta^2-\gamma^*\beta+\mathcal{O}\big(\beta^{2/3}\big).$$

A proof sketch

We employ variational estimate with trial functions

$$\hat{\varphi}(s, u) = \chi_\varepsilon(s)\Big(\mathrm{e}^{-\alpha u} - \mathrm{e}^{-2a\alpha + u\alpha}\Big),$$

where χ_ε is a smooth function on $[0, L]$ with the support located in an ε-neighborhood of a point s^* in which $\gamma(s^*) = \gamma^*$.

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We consider functions of the form

$$\chi_\varepsilon(s) := \chi\Big(\frac{s - s^* + \varepsilon}{2\varepsilon}\Big),$$

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Optimizing the bound by choosing $\varepsilon = \beta^{-1/3}$, we get

$$\frac{b^D_{a,\beta}[\hat{\varphi}]}{\|\hat{\varphi}\|^2_{L^2(0,L)}} \le -\Big(\beta + \frac{\gamma^*}{2}\Big)^2 + \mathcal{O}\big(\beta^{2/3}\big).$$

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This proves the result for the ground state eigenvalue $\lambda_1(\beta)$.

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For the higher eigenfunctions we proceed in the same way, using trial functions of the form

$$\hat{\varphi}_j(s,u) = \chi_{\varepsilon,j}(s)\big(\mathrm{e}^{-\alpha u} - \mathrm{e}^{-2a\alpha+u\alpha}\big),$$

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The estimates remain essentially the same, up to the values of the constants involved. By construction, the functions $\chi_{\varepsilon,j}$ with different values of j have disjoint supports, hence $\hat{\varphi}_j$ is orthogonal to $\hat{\varphi}_i$, $i=1,\dots,j-1$, and by the min-max principle the eigenvalue $\lambda_j(\beta)$ has again the stated upper bound. $\square$

Improvements and generalizations

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$$\lambda_j(\beta) = -\beta^2 + E_j(-\Delta_S - \beta(d-1)H + \mathcal{O}\big(\log\beta\big),$$

where $-\Delta_S$ is the Laplace-Beltrami operator on $S := \partial\Omega$ and H is the mean curvature of the boundary, $(d-1)H = \kappa_1 + \cdots + \kappa_{d-1}$.

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The error term can be further improved if $\partial\Omega$ is more regular.

Improvements and generalizations

Note that for $d = 3$ the *difference between the one- and two-sided situation* is seen: in the Robin case the 'effective potential' is given by the mean curvature only, while for Schrödinger operators it is a combination of Gauss and mean curvatures, $K - M^2$.

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Consider $\Omega \subset \mathbb{R}^2$ with a C^∞ smooth boundary, possibly infinite. Suppose that the curvature κ attains its maximum $\kappa_{\max}$ at a unique point, and the maximum is non-degenerate, i.e. $k_2 := -\kappa''(0) > 0$.

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$$\lambda_n(\beta) = -\beta^2 - \kappa_{\max}\beta + (2n-1)\sqrt{\frac{k_2}{2}}|\beta|^{1/2} + \sum_{j=0}^{M} \zeta_{j,n}|\beta|^{\frac{1-j}{4}} + |\beta|^{\frac{1-M}{4}}o(1)$$

A detour from the detour

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$$\Lambda(\Omega, p, \beta) := \inf_{0 \neq u \in W^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p \, \mathrm{d}x - \beta \int_{\partial\Omega} |u|^p \, \mathrm{d}\sigma}{\int_\Omega |u|^p \, \mathrm{d}x}\,.$$

The detour from the detour, continued

We call a domain $\Omega \subset \mathbb{R}^\nu$, $\nu \geq 2$, admissible if

- the boundary $\partial\Omega$ is $C^{1,1}$, i.e. is locally the graph of a function with a Lipschitz gradient
- the principal curvatures of $\partial\Omega$ are essentially bounded
- the map $\partial\Omega \times (0,\delta) \ni (s,t) \mapsto s - tn(s) \in \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) < \delta\}$ is bijective for some $\delta > 0$

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Theorem (Kovařík-Pankrashkin'16)

For any admissible domain $\Omega \subset \mathbb{R}^\nu$, $\nu \geq 2$ and any $p \in (1,\infty)$ we have

$$\Lambda(\Omega, p, \beta) = -(p-1)\beta^{p/(p-1)} - \beta(\nu-1)H_{\max}(\Omega) + o(\beta)$$

as $\beta \to \infty$.

Back to the main topic: weakly bent curves

The strong coupling regime is not the only asymptotic problem the leaky structure can offer. Let us turn to *geometric perturbations*.

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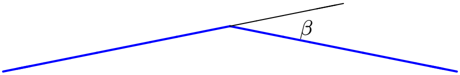

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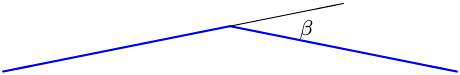

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By the above mentioned result the Hamiltonian has an eigenvalues, a single one for small β, and by a simple scaling argument together with an analogy with bent Dirichlet tubes lead us to conjecture that

$$\lambda(H_{\Gamma_\beta}) = -\frac{1}{4}\alpha^2 + a\beta^4 + o(\beta^4)$$

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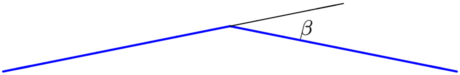

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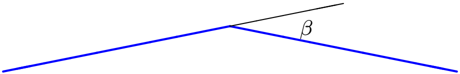

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The question now is (a) what is the coefficient a, and (b) *whether a similar formula holds for more general slightly bent curves*.

Weakly bent curves, continued

Let us first specify the class of curves we shall consider: Γ will be a continuous and piecewise C^2 infinite planar curve without self-intersections parametrized by its arc length, i.e. the graph of a piecewise C^2-smooth function $\Gamma : \mathbb{R} \to \mathbb{R}^2$ such that $|\dot{\gamma}(s)| = 1$. Moreover,

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- there exists a $c \in (0,1)$ such that $|\Gamma(s) - \Gamma(s')| \geq c|s - s'|$ holds for $s, s' \in \mathbb{R}$,
- there are real numbers $s_1 > s_2$ and straight lines Σ_i, $i = 1, 2$, such that Γ coincides with Σ_1 for $s \geq s_1$ and with Σ_2 for $s \leq s_2$,
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- one-sided limits of $\dot\gamma$ exists at the points where the function $\ddot\gamma$ is discontinuous.

In particular, the signed curvature $k(s) = \dot\gamma_2(s)\ddot\gamma_1(s) - \dot\gamma_1(s)\ddot\gamma_2(s)$ is piecewise continuous and the one-sided limits of $\dot\gamma$, i.e. *tangent vectors* to the curve at the points of discontinuity exist. We denote them as $\Pi = \{p_i\}_{i=1}^{\sharp\Pi}$ and shall speak of them as of vertices. Consequently, Γ consists of $\sharp\Pi + 1$ simple arcs or *edges*, each having as its endpoints one or two of the vertices.

Weakly bent curves, continued

The curvature integral describes *bending* of the curve. Specifically, the angle between the tangents at the points $\Gamma(s)$ and $\Gamma(s')$ equals

$$\phi(s,s') = \sum_{p_i\in(s,s')} c(p_i) + \int_{(s,s')\setminus\Pi} k(\zeta)\,\mathrm{d}\zeta\,,$$

where $c(p_i) \in (0,\pi)$ is the exterior angle of the two adjacent edges of Γ meeting at the vertex p_i.

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Alternatively, we can understand $\phi(s,s')$ as the integral over the interval (s,s') of $\tilde{k}:\ \tilde{k}(s) = k(s) + \sum_{p\in\Pi} c(p)\,\delta(s-p)$. By assumption k, $\tilde{k}$ are compactly supported, thus $\phi(s,s')$ has the same value for all $s < s_2$ and $s_1 < s'$ which we shall call the *total bending*.

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One can reconstruct Γ from $\tilde{k}$, uniquely up to Euclidean transformations,

$$\Gamma(s) = \left(\int_0^s \cos\phi(u,0)\,\mathrm{d}u\,, \int_0^s \sin\phi(u,0)\,\mathrm{d}u \right).$$

Weakly bent curves, continued

Now we introduce the one-parameter family of 'scaled' curves Γ_β,

$$\Gamma_\beta(s) = \left(\int_0^s \cos \beta\phi(u,0)\,\mathrm{d}u\,, \int_0^s \sin \beta\phi(u,0))\,\mathrm{d}u \right), \quad |\beta| \in (0,1]\,.$$

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Next we define an integral operator $A:\ L^2(\mathbb{R}) \to L^2(\mathbb{R})$ through its kernel,

$$\mathcal{A}(s,s') := \frac{\alpha^4}{32\pi} K_0' \left(\frac{\alpha}{2}|s-s'| \right) \left(|s-s'|^{-1} \left(\int_{s'}^s \phi \right)^2 - \int_{s'}^s \phi^2 \right).$$

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Lemma

Under the stated assumptions, we have $\int_{\mathbb{R}\times\mathbb{R}} \mathcal{A}(s,s')\, \mathrm{d}s\, \mathrm{d}s' < \infty$.

Weakly bent curves, the result

Now we are in position to state the weak-bending result.

Theorem (E-Kondej'16)

There is a $\beta_0 > 0$ *such that for any* $\beta \in (-\beta_0, 0) \cup (0, \beta_0)$ *the operator* H_{Γ_β} *has a unique eigenvalue* $\lambda(H_{\Gamma_\beta})$ *which admits the asymptotic expansion*

$$\lambda(H_{\Gamma_\beta}) = -\frac{\alpha^2}{4} - \left(\int_{\mathbb{R}\times\mathbb{R}} \mathcal{A}(s, s')\, \mathrm{d}s\, \mathrm{d}s' \right)^2 \beta^4 + o(\beta^4)\,.$$

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Proof is laborious but the idea is simple. It is based again on application of the Birman-Schwinger principle which says that

$$-\kappa^2 \in \sigma_{\mathrm{d}}(H_{\Gamma_\beta}) \quad \Leftrightarrow \quad \ker(I - \alpha Q_{\Gamma_\beta}(\kappa)) \neq \emptyset\,,$$

where $Q_{\Gamma_\beta}(\kappa)$ is the integral operator with the kernel

$$\mathcal{Q}_{\Gamma_\beta}(\kappa; s, s') = \frac{1}{2\pi} K_0(\kappa|\gamma_\beta(s) - \gamma_\beta(s')|)\,;$$

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Theorem (E-Kondej'16)

There is a $\beta_0 > 0$ *such that for any* $\beta \in (-\beta_0, 0) \cup (0, \beta_0)$ *the operator* H_{Γ_β} *has a unique eigenvalue* $\lambda(H_{\Gamma_\beta})$ *which admits the asymptotic expansion*

$$\lambda(H_{\Gamma_\beta}) = -\frac{\alpha^2}{4} - \left(\int_{\mathbb{R}\times\mathbb{R}} \mathcal{A}(s,s')\, \mathrm{d}s\, \mathrm{d}s' \right)^2 \beta^4 + o(\beta^4)\,.$$

Proof is laborious but the idea is simple. It is based again on application of the Birman-Schwinger principle which says that

$$-\kappa^2 \in \sigma_\mathrm{d}(H_{\Gamma_\beta}) \quad \Leftrightarrow \quad \ker(I - \alpha Q_{\Gamma_\beta}(\kappa)) \neq \emptyset\,,$$

where $Q_{\Gamma_\beta}(\kappa)$ is the integral operator with the kernel

$$\mathcal{Q}_{\Gamma_\beta}(\kappa; s, s') = \frac{1}{2\pi} K_0(\kappa|\gamma_\beta(s) - \gamma_\beta(s')|)\,;$$

moreover, we have $\dim\ker(H_{\Gamma_\beta} + \kappa^2) = \dim\ker(I - \alpha Q_{\Gamma_\beta}(\kappa))$.

Weakly bent curves, continued

One has to compare with the Birman-Schwinger operator corresponding to the *straight line* which has the kernel $K_0\left(\frac{\kappa}{2}|s-s'|\right)$ in the vicinity of the point $\kappa = \frac{1}{2}\alpha$ corresponding to threshold of the essential spectrum.

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Let us return to the *broken-line example*: in this case $\mathcal{A}(s,s')$ can be found easily, it vanishes if s, s' have the same sign, being otherwise

$$\mathcal{A}(s,s') = \frac{\alpha^4}{32\pi} K_0'\left(\frac{\alpha}{2}|s-s'|\right) \frac{|ss'|}{|s-s'|}\, \chi_\Omega(s,s')\,,$$

where $\chi_\Omega(\cdot,\cdot)$ is the characteristic function of the set Ω, the union of the second and fourth quadrant.

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where $\chi_\Omega(\cdot,\cdot)$ is the characteristic function of the set Ω, the union of the second and fourth quadrant. The integral of $\mathcal{A}(s,s')$ over the both variable can be computed explicitly giving

$$\frac{-\frac{1}{4}\alpha^2-\lambda(H_{\Gamma_\beta})}{-\frac{1}{4}\alpha^2}=-\frac{1}{9\pi^2}\beta^4+o(\beta^4)\,.$$

Systems with infinite discrete spectrum

One can encounter still another type of asymptotic formulæ in situations when $\sharp\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma} = \infty$. The eigenvalues then typically *accumulate at the bottom of the essential spectrum* and one can ask how fast this accumulation proceeds for a given geometry.

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The first question is whether an infinite discrete spectrum may exist. In examples such as the the *broken line in the plane* Γ_β considered above it is not the case: the number of the bound state can be made large for a sharp break, $\pi - \beta$ sufficiently small, but it remains finite.

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Nevertheless, examples of infinite discrete spectrum exist. This happens, for instance if $d = 3$ and $\Gamma = \mathcal{C}_\theta$ is the *conical surface* of the opening angle 2θ, in other words

$$\mathcal{C}_\theta := \left\{(x,y,z) \in \mathbb{R}^3 :\ z := \cot(\theta)\sqrt{x^2+y^2}\right\}, \quad \theta \in (0, \tfrac{1}{2}\pi)\,.$$

The conical layer spectrum

Theorem (Behrndt-E-Lotoreichik'14)

For any $\theta \in (0, \frac{1}{2}\pi)$ *and* $\alpha > 0$ *the essential spectrum of the operator* $H_{\alpha,\mathcal{C}_\theta}$ *is* $[-\frac{1}{4}\alpha^2, \infty)$*, the discrete spectrum is infinite and accumulates to* $-\frac{1}{4}\alpha^2$.

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Proof sketch: The argument proceeds in several steps

- Using the *cylindrical symmetry* one can perform a partial-wave decomposition and observe that only the s-wave component is important, the spectra of the components with nonzero angular momentum are contained in $[-\frac{1}{4}\alpha^2, \infty)$.
- The problem is thus reduced to two dimensions in a halfplane, $(r, z) \in \mathbb{R}_+ \times \mathbb{R}$, using reduced wave functions, $\psi(r, \varphi, z) = \frac{\omega(r,z)}{\sqrt{2\pi r}}$. We introduce rotated coordinates, s along the halfline representing the surface and t perpendicular to it.

Proof sketch, continued

- To check that $\sigma_{\mathrm{ess}}(H_{\alpha,\mathcal{C}_\theta}) \supset [-\frac14\alpha^2,\infty)$ we construct a suitable Weyl sequence. It can be done, e.g., using

$$\omega_{n,p}(s,t) := \frac{1}{\sqrt{n}}\Big(\chi_1\big(\tfrac{s}{n}\big)\exp(\mathrm{i}ps)\Big)\Big(\chi_2\big(\tfrac{t}{n}\big)\exp\big(-\tfrac{\alpha}{2}|t|\big)\Big)\,,$$

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- The check the opposite inclusion, one uses Neumann bracketing taking a symmetric surface *layer neighbourhood* of the width $2\sqrt{n}$ cut at a distance from the cone tip, $s > n$. Its complement does not contribute to the essential spectrum; choosing n large enough, one can make the influence of the term $-(4r(z))^{-1}$ in the Hamiltonian small and to prove in this way that $\inf\sigma_{\mathrm{ess}}(H_{\alpha,\mathcal{C}_\theta}) \ge -\frac14\alpha^2 - \varepsilon$ holds for any $\varepsilon > 0$ which yields the result.

Proof sketch, continued

- For the discrete spectrum part we chose suitable trial functions, e,g.,

$$\omega_n(s,t) := \frac{1}{n}\chi_1\big(\tfrac{s-n}{n^2}\big)\chi_2\big(\tfrac{t}{\sqrt{n}}\big)\exp\big(-\tfrac{\alpha}{2}|t|\big) \in H_0^1(\mathbb{R}_+^2)\,,$$

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- Choosing the indices n appropriately, we can construct a sequence of functions with disjoint supports and prove in this way that the discrete spectrum is infinite and the eigenvalues λ_k satisfy

$$\lambda_k \le -\frac{\alpha^2}{4} - \frac{\gamma(\theta)}{n_k^4}\,, \quad k\in\mathbb{N}\,,$$

  where $\gamma(\theta) > 0$ with $n_{k+1} := n_k^2 + n_k$ and $n_1 = N$ with $N \in \mathbb{N}$ sufficiently large. □

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Remarks: • The claim remains true if the cone is *locally deformed*.

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Remarks: • The claim remains true if the cone is *locally deformed*.

• $\sigma_{\mathrm{disc}}(H_{\alpha,\mathcal{C}_\theta}) = \emptyset$ holds if $d\ge 4$, cf. [Lotoreichik–Ourmières-Bonafos'16].

Accumulation asymptotics

We denote conventionally $\mathcal{N}_E(T) = \sharp\{k \in \mathbb{N} :\ \lambda_k(T) < E\}$, i.e. the counting function of eigenvalues of the operator T below the threshold of its essential spectrum.

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For the conical layer the discrete spectrum accumulates as

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Proof idea is to estimate the discrete spectrum from above and below using a suitable one-dimensional operator, in this case $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{4\sin^2\theta}\frac{1}{x^2}$ on the interval $(1,\infty)$ the spectral asymptotics of which is known.

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Open questions

In my view, the main challenge concerns the strong-coupling behavior in situations with *less regularity*, in the first place such a behavior for Hamiltonians of *branched leaky graphs.*

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The 'one-sided' case is different: recall also that if the 'attractive' Robin billiard has corners of angles $2\theta_j$, the ground state behaves in the limit $\beta \to \infty$ as $-\beta^2 \max_j \left(\sin^{-2} \theta_j \right)$ – cf. [Levitin-Parnovski'08]. Tunneling between the corners was discussed recently by Helffer and Pankrashkin.

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The talk was based, in particular, on

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as well as the other papers mentioned in the course of the presentation.

It remains to say

Thank you for your attention!