Inequalities for means of chords and related isoperimetric problems

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Talk overview

Motivation: some classical and less classical isoperimetric problems for Schrödinger operators
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Another motivation coming from electrostatics: what shape will a charged loop take?
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- **A discrete analogue** of our problem
- **Summary and outlook**
Motivation

Isoperimetric problems are traditional in mathematical physics. Recall, e.g., the Faber-Krahn inequality for the Dirichlet Laplacian $-\Delta^M_D$ in a compact $M \subset \mathbb{R}^2$: among all regions with a fixed area the ground state is uniquely minimized by the circle,

$$\inf \sigma(-\Delta^M_D) \geq \pi j_{0,1}^2 |M|^{-1};$$

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Another classical example is the PPW conjecture proved by Ashbaugh and Benguria: in the 2D situation we have

$$\frac{\lambda_2(M)}{\lambda_1(M)} \leq \left( \frac{j_{1,1}}{j_{0,1}} \right)^2$$
Notice that topology is important

If \( M \) is not simply connected, rotational symmetry may again lead to an extremum but its nature can be different. Recall a \textit{a strip of fixed length and width} [E.-Harrell-Loss’99]

\[
\text{ground state of } \begin{array}{c}
\includegraphics[width=0.2\textwidth]{strip.png}
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whenever the strip is not a circular annulus.
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Another example is a *circular obstacle in circular cavity* [Harrell-Kröger-Kurata’01]

\[
\text{ground state of } \begin{array}{c}
\includegraphics[width=0.2\textwidth]{circular_obstacle}
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\]

whenever the obstacle is off center
Singular Schrödinger operators

Topology loses meaning when the confinement is due to a potential. For simplicity, we suppose is a singular one,

\[ H_{\alpha, \Gamma} = -\Delta - \alpha \delta(x - \Gamma), \quad \alpha > 0, \]

in \( L^2(\mathbb{R}^2) \), where \( \Gamma \) is a loop in the plane; we suppose that it has no zero-angle self-intersections.
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\( H_{\alpha, \Gamma} \) can be naturally associated with the quadratic form,

\[ \psi \mapsto \|\nabla \psi\|_{L^2(\mathbb{R}^2)}^2 - \alpha \int_{\Gamma} |\psi(x)|^2 \, dx, \]

which is closed and below bounded in \( W^{1,2}(\mathbb{R}^2) \); the second term makes sense in view of Sobolev embedding. This definition also works for various “wilder” sets \( \Gamma \)
Definition by boundary conditions

Since $\Gamma$ is *piecewise smooth* with *no cusps* we can use an *alternative definition* by boundary conditions: $H_{\alpha, \Gamma}$ acts as $-\Delta$ on functions from $W^{2,1}_{\text{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

$$\left. \frac{\partial \psi}{\partial n} (x) \right|_+ - \left. \frac{\partial \psi}{\partial n} (x) \right|_- = -\alpha \psi(x)$$
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**Remarks:**

- This definition has an illustrative meaning which corresponds to a $\delta$ potential in the cross cut of $\Gamma$.
- Using the form associated with $H_{\alpha, \Gamma}$ one can check directly that $\sigma_{\text{disc}}(H_{\alpha, \Gamma})$ **is not void** for any $\alpha > 0$; one has, of course, $\sigma_{\text{ess}}(H_{\alpha, \Gamma}) = [0, \infty)$. We will ask about $\Gamma$ of a fixed length which **maximizes the ground state**.
Charged loops

Let us mention another problem which comes from classical electrostatics and at a glance it has a little in common with the quantum mechanical question posed above.
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Let $\Gamma : [0, L] \rightarrow \mathbb{R}^3$ be a smooth loop and suppose that it is homogeneously charged and non-conducting. We ask about the shape which it will take in absence of external forces, i.e. about minimum of the potential energy of the Coulombic repulsion.

Remark: The latter has to be renormalized. The question makes sense because the divergent factor comes from the short-distance behavior being shape-independent.
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We are going to show that both the mentioned problems reduce essentially to *the same geometric question*.
Inequalities for $L^p$ norms of chords

It is convenient to work in an arbitrary dimension $d \geq 2$. Let $\Gamma$ be a piecewise differentiable function $\Gamma : [0, L] \to \mathbb{R}^d$ such that $\Gamma(0) = \Gamma(L)$ and $|\dot{\Gamma}(s)| = 1$ for any $s \in [0, L]$. Consider chords corresponding to a fixed arc length $u \in (0, \frac{1}{2}L]$; we are interested in the inequalities

$$C^p_L(u) : \quad c^p_{\Gamma}(u) := \int_0^L |\Gamma(s+u) - \Gamma(s)|^p \, ds \leq \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L}, \quad p > 0$$

$$C^{-p}_L(u) : \quad c^{-p}_{\Gamma}(u) := \int_0^L |\Gamma(s+u) - \Gamma(s)|^{-p} \, ds \geq \frac{\pi^p L^{1-p}}{\sin^p \frac{\pi u}{L}}, \quad p > 0$$
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The right sides correspond to the maximally symmetric case, the planar circle. It is clear that the inequalities are invariant under scaling, so without loss of generality we may fix the length, say, to $L = 2\pi$. Notice also that for $p = 0$ the inequalities $C^p_L(u)$ and $C^{-p}_L(u)$ turn into trivial identities.
Simple properties

Using convexity of $x \mapsto x^\alpha$ in $(0, \infty)$ for $\alpha > 1$ we get

**Proposition:** $C^p_L(u) \Rightarrow C^{p'}_L(u)$ if $p > p' > 0$
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The norm can be expressed through curvature of $\Gamma$. Using then a Fourier analysis, one can prove

**Proposition [E’05b]:** If $\Gamma$ is $C^2$, the inequality $C^2_L(u)$, and thus also $C^p_L(u)$ for $|p| \leq 2$, holds locally.
The global result

**Theorem** [Lükoš’66; Abrams et al.’03; E-Harrell-Loss’05]: Let $\Gamma$ be piecewise $C^1$ with no cusps. Then $C^2_L(u)$ is valid for any $u \in (0, \frac{1}{2}L]$, and the inequality is strict unless $\Gamma$ is a planar circle.
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**Theorem** [Lükő’66; Abrams et al.’03; E-Harrell-Loss’05]: Let $\Gamma$ be piecewise $C^1$ with no cusps. Then $C^2_L(u)$ is valid for any $u \in (0, \frac{1}{2}L]$, and the inequality is strict unless $\Gamma$ is a planar circle.

**Proof:** Without loss of generality we put $L = 2\pi$ and write

$$\Gamma(s) = \sum_{0 \neq n \in \mathbb{Z}} c_n e^{i n s}$$

with $c_n \in \mathbb{C}$. Since $\Gamma(s) \in \mathbb{R}^d$ the coefficients have to satisfy $c_{-n} = \bar{c}_n$; the absence of $c_0$ can be always achieved by a choice of the origin of the coordinate system.

In view of the Weierstrass theorem and continuity of the functional in question, we may suppose that $\Gamma$ is $C^2$, apart of the last part of the theorem.
Proof, continued

If $\Gamma$ is $C^2$ its derivative is a sum of the uniformly convergent Fourier series

$$\dot{\Gamma}(s) = i \sum_{0 \neq n \in \mathbb{Z}} n c_n e^{ins}$$
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By assumption, $|\dot{\Gamma}(s)| = 1$, and hence from the relation

$$2\pi = \int_0^{2\pi} |\dot{\Gamma}(s)|^2 \, ds = \int_0^{2\pi} \sum_{0 \neq m \in \mathbb{Z}} \sum_{0 \neq n \in \mathbb{Z}} n m c_m^* \cdot c_n e^{i(n-m)s} \, ds,$$

where $c_m^*$ denotes the row vector $(\bar{c}_{m,1}, \ldots, \bar{c}_{m,d})$ and dot marks the inner product in $\mathbb{C}^d$, we infer that

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1$$
Proof, continued

In a similar way we can rewrite the right-hand side expression of $C_{2\pi}(u)$ using the Parseval relation as

$$\int_0^{2\pi} \left| \sum_{0 \neq n \in \mathbb{Z}} c_n (e^{inu} - 1) e^{ins} \right|^2 \, ds = 8\pi \sum_{0 \neq n \in \mathbb{Z}} |c_n|^2 \left( \sin \frac{nu}{2} \right)^2$$
Proof, continued

In a similar way we can rewrite the right-hand side expression of $C_{2\pi}^2(u)$ using the Parseval relation as

$$\int_0^{2\pi} \left| \sum_{0 \neq n \in \mathbb{Z}} c_n (e^{inu} - 1) e^{ins} \right|^2 \, ds = 8\pi \sum_{0 \neq n \in \mathbb{Z}} |c_n|^2 \left( \sin \frac{nu}{2} \right)^2$$

Thus the sought inequality is equivalent to

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 \left( \frac{\sin \frac{nu}{2}}{n \sin \frac{u}{2}} \right)^2 \leq 1$$

and it is sufficient to prove that $|\sin nx| \leq n \sin x$ holds for all positive integers $n$ and all $x \in (0, \frac{1}{2}\pi]$. 
Proof, continued

We use induction. The claim is valid for $n = 1$ and

$$(n + 1) \sin x \mp \sin (n + 1)x = n \sin x \mp \sin nx \cos x + \sin x (1 \mp \cos nx),$$

so if it holds for $n$, the sum of the first two terms at the rhs is non-negative, and the same is clearly true for the last one.
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We also see that if $|\sin nx| < n \sin x$ the inequality is strict for $n + 1$ as well. Since this is true for for $n = 2$, equality can occur only for $n = 1$. Hence $C^2_{2\pi}(u)$ is strict unless $c_n = 0$ for $|n| \geq 2$, being saturated only if the $j$th projection of $\Gamma$ equals

$$\Gamma_j(s) = 2|c_{1,j}| \cos(s + \arg c_{1,j}).$$

Furthermore, $|\dot{\Gamma}(s)| = 1$ can be true only if there is a basis in $\mathbb{R}^d$ where $c_{1,1} = i c_{1,2} = \frac{1}{2}$ and $c_{1,j} = 0$ for $j = 3, \ldots, d$, in other words, if $\Gamma$ is a planar circle
Proof, conclusion

It remains to check that the inequality cannot be saturated for a curve $\Gamma$ that is not $C^2$, so that the sum $\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2$ diverges
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This would require

$$\sum_{1 \leq n \leq N} n^2 |c_n|^2 \left( \frac{\sin \frac{nu}{2}}{n \sin \frac{u}{2}} \right)^2 \frac{1}{\sum_{1 \leq n \leq N} n^2 |c_n|^2} \to 1$$

as $N \to \infty$. This is impossible, however, because the sum in the numerator is bounded by $\sec^2 \frac{u}{2} \sum_{1 \leq n \leq N} |c_n|^2$ so it has a finite limit; this concludes the proof. □
Application to charged loops

Let $\Gamma$ be a closed $C^2$ curve in $\mathbb{R}^3$, to be compared with a planar circle. The energy cost of such a deformation is $q^2 \delta(\Gamma)$, where

\[
\delta(\Gamma) := 2 \int_0^{L/2} du \int_0^L ds \left[ |\Gamma(s+u) - \Gamma(s)|^{-1} - \frac{\pi}{L} \csc \frac{\pi u}{L} \right]
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and $q$ is the charge density along the loop.
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**Corollary:** $\delta(\Gamma)$ is finite and non-negative; it is zero if and only if $\Gamma = C_L$, up to Euclidean equivalence.

**Proof:** The integrand is $\geq 0$ by $C_L^{-1}(u)$, strictly so if $\Gamma \neq C_L$. Moreover, we have $|\Gamma(s+u) - \Gamma(s)|^{-1} = u^{-1} + O(1)$ with the error dependent on curvature and torsion of $\Gamma$ but uniform in $s$, hence the integral converges. □
Application to leaky loops

Let us turn to the singular Schrödinger operators $H_{\alpha, \Gamma} = -\Delta - \alpha \delta(x - \Gamma)$. We have mentioned that the discrete spectrum is nonempty and finite, in particular,

$$\epsilon_1 \equiv \epsilon_1(\alpha, \Gamma) := \inf \sigma \left( H_{\alpha, \Gamma} \right) < 0$$
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**Theorem [E’05b]:** Let \( \Gamma : [0, L] \rightarrow \mathbb{R}^2 \) have the indicated properties; then for any fixed \( \alpha > 0 \) and \( L > 0 \) the ground state \( \epsilon_1(\alpha, \Gamma) \) is **globally uniquely maximized by the circle** of radius \( L/2\pi \).

**Proof** is based on the generalized Birman-Schwinger principle, plus symmetry and convexity arguments.
We employ the generalized Birman-Schwinger principle [BEKŠ’94]. One starts from the free resolvent $R^k_0$ which is an integral operator in $L^2(\mathbb{R}^2)$ with the kernel

$$G_k(x-y) = \frac{i}{4} H_0^{(1)}(k|x-y|)$$
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Then we introduce embedding operators associated with $R^k_0$ for measures $\mu, \nu$ which are the Dirac measure $m$ supported by $\Gamma$ and the Lebesgue measure $d\,x$ on $\mathbb{R}^2$; by $R^k_{\nu, \mu}$ we denote the integral operator from $L^2(\mu)$ to $L^2(\nu)$ with the kernel $G_k$, i.e. we suppose that

$$R^k_{\nu, \mu} \phi = G_k * \phi \mu$$

holds $\nu$-a.e. for all $\phi \in D(R^k_{\nu, \mu}) \subset L^2(\mu)$
**Proposition [BEKŠ’94, Posilicano’04]:** (i) There is $\kappa_0 > 0$ s.t. $I - \alpha R^k_{m,m}$ on $L^2(m)$ has a bounded inverse for $\kappa \geq \kappa_0$

(ii) Let $\text{Im} \ k > 0$ and $I - \alpha R^k_{m,m}$ be invertible with

$$R^k := R^k_0 + \alpha R^k_{dx,m}[I - \alpha R^k_{m,m}]^{-1} R^k_{m,dx}$$

from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ everywhere defined. Then $k^2$ belongs to $\rho(H_{\alpha,\Gamma})$ and $(H_{\alpha,\Gamma} - k^2)^{-1} = R^k$

(iii) $\dim \ker(H_{\alpha,\Gamma} - k^2) = \dim \ker(I - \alpha R^k_{m,m})$ for $\text{Im} \ k > 0$

(iv) an ef of $H_{\alpha,\Gamma}$ associated with $k^2$ can be written as

$$\psi(x) = \int_0^L \int_0^x R^k_{dx,m}(x, s) \phi(s) \, ds,$$

where $\phi$ is the corresponding ef of $\alpha R^k_{m,m}$ with the ev one
BS reformulation, continued

Putting $k = i\kappa$ with $\kappa > 0$ we look thus for solutions to the integral-operator equation

$$\mathcal{R}_{\alpha, \Gamma}^\kappa \phi = \phi, \quad \mathcal{R}_{\alpha, \Gamma}^\kappa (s, s') := \frac{\alpha}{2\pi} K_0 (\kappa |\Gamma(s) - \Gamma(s')|),$$

on $L^2([0, L])$. The function $\kappa \mapsto \mathcal{R}_{\alpha, \Gamma}^\kappa$ is strictly decreasing in $(0, \infty)$ and $\|\mathcal{R}_{\alpha, \Gamma}^\kappa\| \to 0$ as $\kappa \to \infty$, hence we seek the point where the largest ev of $\mathcal{R}_{\alpha, \Gamma}^\kappa$ crosses one.
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We observe that this ev is simple, since $\mathcal{R}_{\alpha, \Gamma}^\kappa$ is positivity improving and ergodic. The ground state of $H_{\alpha, \Gamma}$ is, of course, also simple. Using its rotational symmetry and the claim (iv) of the Proposition we find that the respective eigenfunction of $\mathcal{R}_{\alpha, \Gamma}^{\tilde{\kappa}_1}$ corresponding to the unit eigenvalue is constant; we can choose it as $\tilde{\phi}_1(s) = L^{-1/2}$. 
Then we have

$$\max \sigma(\mathcal{R}_{\tilde{k}_1}) = (\tilde{\phi}_1, \mathcal{R}_{\tilde{k}_1,\mathcal{C}}\tilde{\phi}_1) = \frac{1}{L} \int_0^L \int_0^L \mathcal{R}_{\tilde{k}_1,\mathcal{C}}(s, s') \, ds \, ds'$$

and on the other hand, for the same quantity referring to a general $\Gamma$ a simple variational estimate gives

$$\max \sigma(\mathcal{R}_{\tilde{k}_1,\Gamma}) \geq (\tilde{\phi}_1, \mathcal{R}_{\tilde{k}_1,\Gamma}\tilde{\phi}_1) = \frac{1}{L} \int_0^L \int_0^L \mathcal{R}_{\tilde{k}_1,\Gamma}(s, s') \, ds \, ds'.$$
BS reformulation, continued

Then we have

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Hence it is sufficient to show that

$$\int_0^L \int_0^L K_0(\kappa|\Gamma(s) - \Gamma(s')|) \, ds \, ds' \geq \int_0^L \int_0^L K_0(\kappa|\mathcal{C}(s) - \mathcal{C}(s')|) \, ds \, ds'$$

holds for all $\kappa > 0$ and $\Gamma$ in the vicinity of $\mathcal{C}$.
By a simple change of variables the claim is equivalent to positivity of the functional

\[ F_\kappa(\Gamma) := \int_0^{L/2} du \int_0^L ds \left[ K_0(\kappa|\Gamma(s+u) - \Gamma(s)|) - K_0(\kappa|\mathcal{C}(s+u) - \mathcal{C}(s)|) \right]; \]

the \( s \)-independent second term is equal to \( K_0\left(\frac{\kappa L}{\pi} \sin \frac{\pi u}{L}\right) \)
Convexity argument

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the \( s \)-independent second term is equal to \( K_0(\frac{\kappa L}{\pi} \sin \frac{\pi u}{L}) \)

The (strict) convexity of \( K_0 \) yields by means of Jensen inequality the estimate

\[ \frac{1}{L} F_\kappa(\Gamma) \geq \int_0^{L/2} \left[ K_0 \left( \frac{\kappa}{L} \int_0^L |\Gamma(s+u) - \Gamma(s)| ds \right) - K_0 \left( \frac{\kappa L}{\pi} \sin \frac{\pi u}{L} \right) \right] du , \]

where the inequality is sharp unless \( \int_0^L |\Gamma(s+u) - \Gamma(s)| ds \) is independent of \( s \)
Finally, we observe that $K_0$ is decreasing in $(0, \infty)$, hence it is sufficient to check the inequality

$$\int_0^L |\Gamma(s+u) - \Gamma(s)| \, ds \leq \frac{L^2}{\pi} \sin \frac{\pi u}{L}$$

for all $u \in (0, \frac{1}{2}L]$ and furthermore, to show that it is strict unless $\Gamma$ is a circle.
Finally, we observe that $K_0$ is decreasing in $(0, \infty)$, hence it is sufficient to check the inequality

$$
\int_0^L |\Gamma(s+u) - \Gamma(s)| \, ds \leq \frac{L^2}{\pi} \sin \frac{\pi u}{L}
$$

for all $u \in (0, \frac{1}{2}L]$ and furthermore, to show that it is strict unless $\Gamma$ is a circle.

In this way our problem is reduced to the $C^1_L(u)$ inequality which follows from $C^2_L(u)$ proved above. □
For which $p$ do the inequalities hold?

It is natural to expect that the inequality $C^p_L(u)$ may be invalid for large enough $p$. A “stadium-perimeter” example in [E-Harrell-Loss’05] shows it is the case for $p \gtrsim 3.15295$. 
For which \( p \) do the inequalities hold?

It is natural to expect that the inequality \( C^p_L(u) \) may be invalid for large enough \( p \). A “stadium-perimeter” example in [E-Harrell-Loss’05] shows it is the case for \( p \gtrsim 3.15295 \).

To find critical \( p \) notice that for a \( C^2 \)-smooth \( \Gamma \) we have

\[
c^p_\Gamma(u) = \int_0^L ds \left[ \int_s^{s+u} ds' \int_s^{s+u} ds'' \cos \left( \int_{s'}^{s''} \gamma(\tau) d\tau \right) \right]^{p/2},
\]

where \( \gamma := \dot{\Gamma}_2 \ddot{\Gamma}_1 - \dot{\Gamma}_1 \ddot{\Gamma}_2 \) is signed curvature of \( \Gamma \). Recall that

\[
\Gamma(s) = \left( \int_0^s \cos \beta(t) \, dt, \int_0^s \sin \beta(t) \, dt \right),
\]

where \( \beta(s) := \int_0^s \gamma(t) \, dt \) is the arc bending.
Variations of the circle

Let us inspect the functional $\Gamma \mapsto c_\Gamma^p(u)$ for curves

$$\gamma(s) = \frac{2\pi}{L} + \varepsilon g(s),$$

where $g$ is continuous and $L$-periodic, so we can write

$$g(s) = a_0 + \sum_{n=1}^{\infty} a_n \sin \left( \frac{2\pi ns}{L} \right) + b_n \cos \left( \frac{2\pi ns}{L} \right)$$

with $\{a\}, \{b\} \in \ell^2$, and $\varepsilon$ is small, i.e. $\varepsilon \|g\|_{\infty} \ll 1$. 
Variations of the circle

Let us inspect the functional $\Gamma \mapsto c^p_\Gamma (u)$ for curves

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where $g$ is continuous and $L$-periodic, so we can write

$$g(s) = a_0 + \sum_{n=1}^{\infty} a_n \sin \left( \frac{2\pi n s}{L} \right) + b_n \cos \left( \frac{2\pi n s}{L} \right)$$

with $\{a\}, \{b\} \in \ell^2$, and $\varepsilon$ is small, i.e. $\varepsilon \|g\|_{\infty} \ll 1$.

Recall that the proof in [E’05b] used the Fourier expansion to check that circle is a local minimum for $p = 2$. Let us now compute the first and second Gâteaux derivatives of the map $\Gamma \mapsto c^p_\Gamma (u)$ at the circle for a general $p$. 
Closeness of $\Gamma$

We must make sure that $\Gamma$ is closed (up to higher terms)

**Proposition:** The tangent to $\Gamma \in C^2$ is $L$-periodic iff $a_0 = 0$. Furthermore, $\Gamma(0) = \Gamma(L) + O(\varepsilon^3)$ provided $a_1 = b_1 = 0$ and

$$\sum_{n=2}^{\infty} \frac{b_n b_{n+1} + a_n a_{n+1}}{n(n+1)} = \sum_{n=2}^{\infty} \frac{a_{n+1} b_n - b_{n+1} a_n}{n(n+1)} = 0$$
Closeness of $\Gamma$

We must make sure that $\Gamma$ is closed (up to higher terms).

**Proposition:** The tangent to $\Gamma \in C^2$ is $L$-periodic iff $a_0 = 0$. Furthermore, $\Gamma(0) = \Gamma(L) + \mathcal{O}(\varepsilon^3)$ provided $a_1 = b_1 = 0$ and

$$\sum_{n=2}^{\infty} \frac{b_n b_{n+1} + a_n a_{n+1}}{n(n+1)} = \sum_{n=2}^{\infty} \frac{a_{n+1} b_n - b_{n+1} a_n}{n(n+1)} = 0$$

**Proof** is based on the mentioned “reconstruction” formula in combination with expansions using $b(s) := \int_0^s g(t)dt$, namely

$$\cos \beta(s) = \left(1 - \frac{1}{2} \varepsilon^2 b^2(s)\right) \cos s - \varepsilon b(s) \sin s + \mathcal{O}(\varepsilon^3),$$

$$\sin \beta(s) = \left(1 - \frac{1}{2} \varepsilon^2 b^2(s)\right) \sin s + \varepsilon b(s) \cos s + \mathcal{O}(\varepsilon^3).$$
Gâteaux derivatives

Put $L = 2\pi$. The first derivative in the direction $g$ is

$$D_g c^p_T(u) = -\frac{p}{2} \left[ 4 \sin^2 \frac{u}{2} \right]^{p/2-1} \int_0^{2\pi} \int_s^{s+u} \int_s^{s+u} \int_s^{s''} \sin \left( \int_{s'}^{s''} dt \right) g(\tau) d\tau$$

The integrals are equal to $(4 \sin^2 u + u \sin u) \int_0^L g(\tau) d\tau = 0$ which shows that circle is for every $p > 0$ a critical point.
Gâteaux derivatives

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\[
D_g c^p_T(u) = -\frac{p}{2} \left[ 4\sin^2 \frac{u}{2} \right]^{p/2-1} \int_0^{2\pi} ds \int \frac{s+u}{s} ds' \int \frac{s+u}{s} ds'' \sin \left( \int_{s'}^{s} dt \right) \int g(\tau) d\tau
\]

The integrals are equal to \( (4\sin^2 u + u \sin u) \int_0^L g(\tau) d\tau = 0 \) which shows that circle is for every \( p > 0 \) a critical point

Next, the second Gâteaux derivative \( D^2_g c^p_T(u) \) equals

\[
\frac{p}{2} \left( \frac{p}{2} - 1 \right) \left[ 4\sin^2 \frac{u}{2} \right]^{p/2-2} \int_0^{2\pi} ds \left( \int \frac{s+u}{s} ds' \int \frac{s+u}{s} ds'' \sin(s'' - s') \int g(\tau) d\tau \right)^2
\]

\[
-\frac{p}{2} \left[ 4\sin^2 \frac{u}{2} \right]^{p/2-1} \int_0^{2\pi} ds \int \frac{s+u}{s} ds' \int \frac{s+u}{s} ds'' \cos(s'' - s') \left( \int g(\tau) d\tau \right)^2
\]
Inspecting the second derivative yields the following claim:

**Theorem [E-Fraas-Harrell’07]:** For a fixed $u \in (0, \frac{1}{2}L]$ define

$$p_c(u) := \frac{4 - \cos\left(\frac{2\pi u}{L}\right)}{1 - \cos\left(\frac{2\pi u}{L}\right)},$$

then we have the alternative: for $p > p_c(u)$ the circle is either a saddle point or a local minimum, while for $p < p_c(u)$ it is a local maximum of $\Gamma \mapsto c^p_\Gamma(u)$. In particular, $p_c\left(\frac{1}{2}L\right) = \frac{5}{2}$.
Critical exponent

Inspecting the second derivative yields the following claim:

**Theorem [E-Fraas-Harrell’07]:** For a fixed $u \in (0, \frac{1}{2}L]$ define

$$p_c(u) := \frac{4 - \cos \left( \frac{2\pi u}{L} \right)}{1 - \cos \left( \frac{2\pi u}{L} \right)},$$

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**Remarks:**
(a) we do not discuss the critical case $p = p_c(u)$ when higher derivatives of $c^p_\Gamma(u)$ come into play
(b) It is natural to expect and easy to verify that for $p > p_c$ circle is in fact a saddle point of the functional
Relation between the critical exponent $p_c$ and the arc length $u$ for $L = 2\pi$. The inequalities hold locally in the region I.
Sketch of the proof

Put again $L = 2\pi$. Using Fourier expansion we cast the second derivative given above into the form

$$D^2 g c_T^p(u) = \sum_{n=2}^{\infty} \left( a_n^2 + b_n^2 \right) \frac{2p\pi \sin^{p-2} \left( \frac{u}{2} \right)}{8(n - n^3)^2} p T(n, u, p),$$

where $T(n, u, p)$ is denotes the following expression

$$-\left( 2n^4 - 6n^2 - 2(n^2 - 1)^2 \cos u + (n + 1)^2 \cos(n - 1)u + (n - 1)^2 \cos(n + 1)u \right)$$

$$+ 2(p - 2) \left( -2n \cos \left( \frac{nu}{2} \right) \sin \left( \frac{u}{2} \right) + 2 \cos \left( \frac{u}{2} \right) \sin \left( \frac{nu}{2} \right) \right)^2$$
Put again $L = 2\pi$. Using Fourier expansion we cast the second derivative given above into the form

$$D^2_g c^p \Gamma(u) = \sum_{n=2}^{\infty} \left( a_n^2 + b_n^2 \right) \frac{2^p \pi \sin^{p-2} \left( \frac{u}{2} \right)}{8(n - n^3)^2} \ p T(n, u, p),$$

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$$+ 2(p - 2) \left( -2n \cos \left( \frac{nu}{2} \right) \sin \left( \frac{u}{2} \right) + 2 \cos \left( \frac{u}{2} \right) \sin \left( \frac{nu}{2} \right) \right)^2$$

Since $\sin(u/2) > 0$ for $u \in (0, \pi)$ the sign of each term is determined by that of $T(n, u, p)$. It is straightforward to check that $T(2, u, p) > 0$ for $p > p_c(u)$, hence for $p > p_c(u)$ the circle fails to be a local maximum of $\Gamma \mapsto c^p \Gamma(u)$.
Sketch of the proof

It is easy to see that $p \mapsto T(n, u, p)$ is strictly increasing, hence to prove the other part of the theorem it is sufficient to show that $T(n, u, p_c(u))$ is negative for $n \geq 3$. We define

$$S(n, u) = -(1 - \cos u) T(n, u, p_c(u))$$

and prove that this function is positive for $n \geq 3$. 
It is easy to see that $p \mapsto T(n, u, p)$ is strictly increasing, hence to prove the other part of the theorem it is sufficient to show that $T(n, u, p_c(u))$ is negative for $n \geq 3$. We define

$$S(n, u) = -(1 - \cos u) \, T(n, u, p_c(u))$$

and prove that this function is positive for $n \geq 3$

In the case $n = 3$ the positivity of $S(n, u)$ is rather easily established since

$$S(3, u) = 2 \left(2 \sin \frac{u}{2}\right)^8,$$

while for $n \geq 4$ the same result follows from a series of simple if somewhat tedious estimates. □
A discrete analogue: polymer loops

Consider a problem related to the above one; following [AGHH’88, 05] we can call it a polymer loop
A discrete analogue: polymer loops

Consider a problem related to the above one; following [AGHH’88, 05] we can call it a polymer loop.

It is an extension of the “discrete” problem to a more general class of curves: we take a closed loop $\Gamma$ and consider a class of singular Schrödinger operators in $L^2(\mathbb{R}^d)$, $d = 2, 3$, given formally by the expression

$$H_{\alpha,\Gamma}^N = -\Delta + \tilde{\alpha} \sum_{j=0}^{N-1} \delta \left( x - \Gamma \left( \frac{jL}{N} \right) \right)$$

We are interested in the shape of $\Gamma$ which maximizes the ground state energy provided, of course, that the discrete spectrum of $H_{\alpha,\Gamma}^N$ is non-empty.
A reminder: 2D point interactions

Fixing the site $y_j$ and “coupling constant” $\alpha$ we define them by b.c. which change \textit{locally} the domain of $-\Delta$: we require

$$\psi(x) = -\frac{1}{2\pi} \log |x - y_j| L_0(\psi, y_j) + L_1(\psi, y_j) + \mathcal{O}(|x - y_j|),$$

where the generalized b.v. $L_0(\psi, y_j)$ and $L_1(\psi, y_j)$ satisfy

$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0, \quad \alpha \in \mathbb{R}$$
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$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0, \quad \alpha \in \mathbb{R}$$

For $Y_\Gamma := \{y_j := \Gamma \left(\frac{jL}{N}\right) : j = 0, \ldots, N - 1\}$ we define in this way $-\Delta_{\alpha, Y_\Gamma}$ in $L^2(\mathbb{R}^2)$. It holds $\sigma_{\text{disc}} \left(-\Delta_{\alpha, Y_\Gamma}\right) \neq \emptyset$, i.e.

$$\epsilon_1 \equiv \epsilon_1(\alpha, Y_\Gamma) := \inf \sigma \left(-\Delta_{\alpha, Y_\Gamma}\right) < 0,$$

which is always true in two dimensions – cf. [AGHH’88, 05]
A reminder: 3D point interactions

Similarly, for $y_j$ and “coupling” $\alpha$ we define them by b.c. which change locally the domain of $-\Delta$: we require

$$\psi(x) = \frac{1}{4\pi |x - y_j|} L_0(\psi, y_j) + L_1(\psi, y_j) + O(|x - y_j|),$$

where the b.v. $L_0(\psi, y_j)$ and $L_1(\psi, y_j)$ satisfy again

$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0, \quad \alpha \in \mathbb{R},$$
A reminder: 3D point interactions

Similarly, for \( y_j \) and “coupling” \( \alpha \) we define them by b.c. which change locally the domain of \(-\Delta\): we require

\[
\psi(x) = \frac{1}{4\pi|x - y_j|} \left( L_0(\psi, y_j) + L_1(\psi, y_j) + O(|x - y_j|) \right),
\]

where the b.v. \( L_0(\psi, y_j) \) and \( L_1(\psi, y_j) \) satisfy again

\[
L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0, \quad \alpha \in \mathbb{R},
\]

giving \(-\Delta_{\alpha,Y_\Gamma}\) in \(L^2(\mathbb{R}^3)\). However, \( \sigma_{\text{disc}} (-\Delta_{\alpha,Y_\Gamma}) \neq \emptyset \), i.e.

\[
\epsilon_1 \equiv \epsilon_1(\alpha, Y_\Gamma) := \inf \sigma (-\Delta_{\alpha,Y_\Gamma}) < 0,
\]

is now a nontrivial requirement; it holds only for \( \alpha \) below some critical value \( \alpha_0 \) – cf. [AGHH’88, 05]
A geometric reformulation

By Krein’s formula, the spectral condition is reduced to an algebraic problem. Using \( k = i\kappa \) with \( \kappa > 0 \), we find the ev’s \(-\kappa^2\) of our operator from

\[
\det \Gamma_k = 0 \quad \text{with} \quad (\Gamma_k)_{ij} := (\alpha - \xi^k)\delta_{ij} - (1 - \delta_{ij})g^k_{ij},
\]

where the off-diagonal elements are \( g^k_{ij} := G_k(y_i - y_j) \), or equivalently

\[
g^k_{ij} = \frac{1}{2\pi} K_0(\kappa|y_i - y_j|)
\]

and the regularized Green’s function at the interaction site is

\[
\xi^k = -\frac{1}{2\pi} \left( \ln \frac{\kappa}{2} + \gamma_E \right)
\]
Geometric reformulation, continued

The ground state refers to the point where the \textit{lowest} \ ev
of $\Gamma_{i\kappa}$ vanishes. Using smoothness and monotonicity
of the $\kappa$-dependence we have to check that

$$\min \sigma(\Gamma_{i\tilde{\kappa}}) < \min \sigma(\tilde{\Gamma}_{i\tilde{\kappa}})$$

holds locally for $\Gamma \neq \tilde{\mathcal{P}}_N$, where $-\tilde{\kappa}_1^2 := \epsilon_1(\alpha, \tilde{\mathcal{P}}_N)$.
Geometric reformulation, continued

The ground state refers to the point where the lowest ev of $\Gamma_{i\kappa}$ vanishes. Using smoothness and monotonicity of the $\kappa$-dependence we have to check that

$$\min \sigma(\Gamma_{i\kappa_1}) < \min \sigma(\tilde{\Gamma}_{i\kappa_1})$$

holds locally for $\Gamma \neq \tilde{\mathcal{P}}_N$, where $-\kappa_1^2 := \epsilon_1(\alpha, \tilde{\mathcal{P}}_N)$.

There is a one-to-one relation between an ef $c = (c_1, \ldots, c_N)$ of $\Gamma_{i\kappa}$ at that point and the corresponding ef of $-\Delta_{\alpha,\Gamma}$ given by $c \leftrightarrow \sum_{j=1}^{N} c_j G_{i\kappa}(\cdot - y_j)$, up to normalization. In particular, the lowest ev of $\tilde{\Gamma}_{i\kappa_1}$ corresponds to the eigenvector $\tilde{\phi}_1 = N^{-1/2}(1, \ldots, 1)$; hence the spectral threshold is

$$\min \sigma(\tilde{\Gamma}_{i\kappa_1}) = (\tilde{\phi}_1, \tilde{\Gamma}_{i\kappa_1} \tilde{\phi}_1) = \alpha - \xi^i\kappa_1 - \frac{2}{N} \sum_{i<j} \tilde{g}^i\kappa_1$$
On the other hand, we have $\min \sigma(\Gamma_{i\kappa_1}) \leq (\tilde{\phi}_1, \Gamma_{i\kappa_1} \tilde{\phi}_1)$, and therefore it is sufficient to check that

$$\sum_{i<j} G_{i\kappa}(y_i - y_j) > \sum_{i<j} G_{i\kappa}(\tilde{y}_i - \tilde{y}_j)$$

holds for all $\kappa > 0$ and $\Gamma \neq \tilde{\mathcal{P}}_N$. 
Geometric reformulation, continued

On the other hand, we have \( \min \sigma(\Gamma \kappa_1) \leq (\tilde{\phi}_1, \Gamma \kappa_1 \phi_1) \), and therefore it is sufficient to check that

\[
\sum_{i<j} G_{i\kappa}(y_i - y_j) > \sum_{i<j} G_{i\kappa}(\tilde{y}_i - \tilde{y}_j)
\]

holds for all \( \kappa > 0 \) and \( \Gamma \neq \tilde{\mathcal{P}}_N \). Call \( \ell_{ij} := |y_i - y_j| \) and \( \tilde{\ell}_{ij} := |\tilde{y}_i - \tilde{y}_j| \) and define \( F : (\mathbb{R}_+)^{N(N-3)/2} \to \mathbb{R} \) by

\[
F(\{\ell_{ij}\}) := \sum_{m=2}^{[N/2]} \sum_{|i-j|=m} \left[ G_{i\kappa}(\ell_{ij}) - G_{i\kappa}(\tilde{\ell}_{ij}) \right];
\]

Using the convexity of \( G_{i\kappa}(\cdot) \) for a fixed \( \kappa > 0 \) we get

\[
F(\{\ell_{ij}\}) \geq \sum_{m=2}^{[N/2]} \nu_m \left[ G_{i\kappa}\left( \frac{1}{\nu_m} \sum_{|i-j|=m} \ell_{ij} \right) - G_{i\kappa}(\tilde{\ell}_{1,1+m}) \right],
\]

where \( \nu_n \) is the number of the appropriate chords.
Geometric reformulation, continued

It is easy to see that

\[ \nu_m := \begin{cases} 
N & \ldots \quad m = 1, \ldots, \left[ \frac{1}{2} (N - 1) \right] \\
\frac{1}{2} N & \ldots \quad m = \frac{1}{2} N \quad \text{for } N \text{ even} 
\end{cases} \]

since for an even \( N \) one has to prevent double counting.
Geometric reformulation, continued

It is easy to see that

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N & \ldots \quad m = 1, \ldots, \left[ \frac{1}{2}(N - 1) \right] \\
\frac{1}{2}N & \ldots \quad m = \frac{1}{2}N \quad \text{for } N \text{ even}
\end{cases}$$

since for an even $N$ one has to prevent double counting

Since $G_{i\kappa}(\cdot)$ is also \textit{monotonously decreasing} in $(0, \infty)$, we thus need only to demonstrate that

$$\tilde{\ell}_{1,m+1} \geq \frac{1}{\nu_n} \sum_{|i-j|=m} \ell_{ij}$$

with the sharp inequality for at least one $m$ if $P_N \neq \tilde{P}_N$.

In this way the problem becomes again purely geometric.
“Discrete” chord inequalities

Recall that for $\Gamma : [0, L] \rightarrow \mathbb{R}^2$ we have used the notation

$$y_j := \Gamma \left( \frac{jL}{N} \right), \quad j = 0, 1, \ldots, N - 1;$$
“Discrete” chord inequalities

Recall that for $Γ : [0, L] \rightarrow \mathbb{R}^2$ we have used the notation

$$y_j := Γ \left( \frac{jL}{N} \right), \quad j = 0, 1, \ldots, N - 1;$$

For fixed $L > 0$, $N$ and $m = 1, \ldots, \lfloor \frac{1}{2} N \rfloor$ we consider the following inequalities for $\ell^p$ norms related to the chord lengths, that is, the quantities $Γ \left( \cdot + \frac{jL}{N} \right) - Γ(\cdot)$

$$D^p_{L,N}(m) : \quad \sum_{n=1}^{N} |y_{n+m} - y_n|^p \leq \frac{N^{1-p} L^p \sin^p \frac{\pi m}{N}}{\sin^p \frac{\pi}{N}}, \quad p > 0,$$

$$D^{-p}_{L,N}(m) : \quad \sum_{n=1}^{N} |y_{n+m} - y_n|^{-p} \geq \frac{N^{1+p} \sin^p \frac{\pi}{N}}{L^p \sin^p \frac{\pi m}{N}}, \quad p > 0.$$

The RHS’s correspond to regular planar polygon $\tilde{P}_N$. 

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More on the "discrete" inequalities

In general, the inequalities are not valid for \( p > 2 \) as the example of a rhomboid shows: \( D_{L,4}^p(2) \) is equivalent to

\[
\sin^p \phi + \cos^p \phi \leq 2^{1-(p/2)} \text{ for } 0 < \phi < \pi \text{ which obviously holds for } p \leq 2 \text{ only}
\]
More on the "discrete" inequalities

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**Proposition:** \( D_{L,N}^p(m) \Rightarrow D_{L,N}^{p'}(m) \) if \( p > p' > 0 \) and

\( D_{L,N}^p(m) \Rightarrow D_{L,N}^{-p}(m) \) for any \( p > 0 \)
More on the "discrete" inequalities

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Proposition: \( D_{L,N}^p(m) \Rightarrow D_{L,N}^{p'}(m) \) if \( p > p' > 0 \) and
\( D_{L,N}^p(m) \Rightarrow D_{L,N}^{−p}(m) \) for any \( p > 0 \)

Theorem [E’05c]: The inequality \( D_{L,N}^2(m) \) is valid
More on the "discrete" inequalities

In general, the inequalities are not valid for $p > 2$ as the example of a rhomboid shows: $D_{L,4}^p(2)$ is equivalent to

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**Proposition:** $D_{L,N}^p(m) \Rightarrow D_{L,N}^{p'}(m)$ if $p > p' > 0$ and $D_{L,N}^p(m) \Rightarrow D_{L,N}^{-p}(m)$ for any $p > 0$

**Theorem [E’05c]:** The inequality $D_{L,N}^2(m)$ is valid

**Remark:** By $D_{L,N}^{-1}(m)$ this implies that the unique minimizers of the “discrete” electrostatic problem is the regular planar polygon $\tilde{P}_N$
Global validity of $D_{L,N}^2(m)$

Let us adapt the above proof the “discrete” case. We put $L = 2\pi$ and express $\Gamma$ through its Fourier series,

$$\Gamma(s) = \sum_{0 \neq n \in \mathbb{Z}} c_n e^{in\lambda}$$

with $c_n \in \mathbb{C}^d$; since $\Gamma(s) \in \mathbb{R}^d$ one has to require $c_{-n} = \overline{c_n}$. Again, we can choose $c_0 = 0$ and the normalization condition $\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1$ follows from $|\dot{\Gamma}(s)| = 1$
Global validity of $D^2_{L,N}(m)$

Let us to adapt the above proof the “discrete” case. We put $L = 2\pi$ and express $\Gamma$ through its Fourier series,

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with $c_n \in \mathbb{C}^d$; since $\Gamma(s) \in \mathbb{R}^d$ one has to require $c_{-n} = \overline{c_n}$. Again, we can choose $c_0 = 0$ and the normalization condition $\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1$ follows from $|\dot{\Gamma}(s)| = 1$

On the other hand, the left-hand side of $D^2_{2\pi,N}(m)$ equals

$$\sum_{n=1}^{N} \sum_{0 \neq j, k \in \mathbb{Z}} c_j^* \cdot c_k \left( e^{-2\pi imj/N} - 1 \right) \left( e^{2\pi imk/N} - 1 \right) e^{2\pi in(k-j)/N}$$
Next we change the order of summation and observe that
\[ \sum_{n=1}^{N} e^{2\pi in(k-j)/N} = N \text{ if } j = k \pmod{N} \text{ and zero otherwise}; \]
this allows us to write the last expression as
\[
4N \sum_{l \in \mathbb{Z}} \sum_{\substack{0 \neq j, k \in \mathbb{Z} \ \mid \ j - k = lN}} |j|c_j^* \cdot |k|c_k \left| j^{-1} \sin \frac{\pi mj}{N} \right| \left| k^{-1} \sin \frac{\pi mk}{N} \right| .
\]
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Hence the sought inequality \( D_{2\pi,N}^2(m) \) is equivalent to

\[ \left( d, (A^{(N,m)} \otimes I) d \right) \leq \left( \frac{\pi \sin \frac{\pi m}{N}}{N \sin \frac{\pi}{N}} \right)^2. \]
Global validity, continued

Here the vector $d \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^d$ has the components $d_j := |j|c_j$ and the operator $A^{(N,m)}$ on $\ell^2(\mathbb{Z})$ is defined as

$$A_{jk}^{(N,m)} := \begin{cases} |j^{-1} \sin \frac{\pi mj}{N}| \cdot |k^{-1} \sin \frac{\pi mk}{N}| & \text{if } 0 \neq j, k \in \mathbb{Z}, j - k = lN \\ 0 & \text{otherwise} \end{cases}$$

$A^{(N,m)}$ is obviously bounded because its Hilbert-Schmidt norm is finite; we have to estimate its norm.
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**Remark:** The “continuous” case corresponds formally to $N = \infty$. Then $A^{(N,m)}$ is a multiple of $I$ and it is only necessary to employ $|\sin jx| \leq j \sin x$ for any $j \in \mathbb{N}$ and $x \in (0, \frac{1}{2}\pi]$. Here due to *infinitely many side diagonals* such a simple estimate yields an unbounded Toeplitz-type operator, and one has use the matrix-element decay.
Global validity, continued

For a given $j \neq 0$ and $d \in \ell^2(\mathbb{Z})$ we have

$$\left( A^{(N,m)} d \right)_j = \left| j^{-1} \sin \frac{\pi mj}{N} \right| \sum_{0 \neq k \in \mathbb{Z}} \left| k^{-1} \sin \frac{\pi mk}{N} \right| d_k$$

where $k = j \pmod{N}$.
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The norm $\|A^{(N,m)}d\|$ is then easily estimated by means of Schwarz inequality,

$$\|A^{(N,m)}d\|^2 = \sum_{0 \neq j \in \mathbb{Z}} j^{-2} \sin^2 \frac{\pi mj}{N} \sum_{0 \neq k \in \mathbb{Z}} \left| k^{-1} \sin \frac{\pi mk}{N} \right| d_k$$

$$\leq \sum_{n=0}^{N-1} \sin^4 \frac{\pi mn}{N} S_n^2 \sum_{n + lN \neq 0} |d_{n+lN}|^2$$
Global validity, concluded

Here we have introduced

$$S_n := \sum_{n + lN \neq 0} \frac{1}{(n + lN)^2} = \sum_{l=1}^{\infty} \left\{ \frac{1}{(lN - n)^2} + \frac{1}{(lN - N + n)^2} \right\}$$

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The sought claim, the validity of \( D_{L,N}^2(m) \), then follows from

\[
\sin \frac{\pi m}{N} \sin \frac{\pi r}{N} > \left| \sin \frac{\pi}{N} \sin \frac{\pi mr}{N} \right|, \quad 2 \leq r < m \leq \left[ \frac{1}{2} N \right]
\]

This can be also equivalently written as the inequalities

\[ U_{m-1} \left( \cos \frac{\pi}{N} \right) > \left| U_{m-1} \left( \cos \frac{\pi r}{N} \right) \right| \quad \text{for Chebyshev polynomials of the second kind; they are verified directly} \]
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We have analyzed a class of inequalities with applications to physically interesting isoperimetric problems. Various questions remains open, for instance

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