Reflections on Zeno and anti-Zeno

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Talk overview

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- **Anti-Zeno effect**: what is it?
- **Sufficient conditions** for anti-Zeno effect
Quantum kinematics of decays

Three objects are needed:

- the state space $\mathcal{H}$ of an *isolated system*
- projection $P$ to subspace $P\mathcal{H} \subset \mathcal{H}$ of *unstable system*
- *time evolution* $e^{-iHt}$ on $\mathcal{H}$, not reduced by $P$ for $t > 0$
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Suppose that evolution starts at $t = 0$ from a state $\psi \in PH$ and we perform a *non-decay measurement* at some $t > 0$.

The non-decay probabilities define in this situation the *decay law*, i.e. the function $P : \mathbb{R}_+ \to [0, 1]$ defined by

$$P(t) := \|P e^{-iHt} \psi\|^2;$$

we may also denote it as $P_\psi(t)$ to indicate the initial state.
Repeated measurements

Suppose we perform non-decay measurements at times $t/n, 2t/n \ldots, t$, all with the positive outcome, then the resulting non-decay probability is

$$M_n(t) = P_{\psi}(t/n)P_{\psi_1}(t/n) \cdots P_{\psi_{n-1}}(t/n),$$

where $\psi_{j+1}$ is the normalized projection of $e^{-iHt/n}\psi_j$ on $P\mathcal{H}$ and $\psi_0 := \psi$, in particular, for $\dim P = 1$ we have

$$M_n(t) = (P_{\psi}(t/n))^n$$
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$$M_n(t) = (P_\psi(t/n))^n.$$

Consider the limit of permanent measurement, $n \to \infty$. If $\dim P = 1$ and the one-sided derivative $\dot{P}(0^+)$ vanishes, we find $M(t) := \lim_{n \to \infty} M_n(t) = 1$ for all $t > 0$, or Zeno effect. The same is true if $\dim P > 1$ provided the derivative $\dot{P}_\psi(0^+)$ has such a property for any $\psi \in PH$. 
When does Zeno effect occur?

Recall first a simple old result:

**Theorem [E.-Havlíček, 1973]:** \( \dot{P}_\psi(0+) = 0 \) holds whenever \( \psi \in Q(H) \)
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**Remarks:**

- Naturally, \( M(t) = P(t) \) if the undisturbed decay law is exponential, i.e. \( P(t) = e^{-\Gamma t} \)

- However, \( P(t) = e^{-\Gamma t} \) correspond to a state not belonging to \( Q(H) \). And what is worse, decay law exponentiality requires \( \sigma(H) = \mathbb{R} \)!
A bit of history

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- Its popularity followed the paper [Misra-Sudarshan’77] where the name quantum Zeno effect was coined
- New interest in recent years, in particular, because the effect becomes experimentally accessible in its non-ideal form: lifetime enhancement by measurement
- New mathematical questions, in particular, about Zeno dynamics: what is the time evolution in \( \mathcal{P\mathcal{H}} \) generated by permanent observation?
Assume that $H$ is \textit{bounded from below} and consider the non-trivial situation, $\dim \mathcal{H} > 1$. We ask: does the limit

$$(P e^{-iHt/n} P)^n \longrightarrow e^{-iH_P t}$$

hold as $n \rightarrow \infty$, in which sense, and what is then Zeno dynamics generator, i.e. the operator $H_P$?
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Consider quadratic form $u \mapsto \|H^{1/2} P u\|^2$ with the form domain $D(H^{1/2} P)$ which is closed. By [Chernoff’74] the associated s-a operator, $(H^{1/2} P)^*(H^{1/2} P)$, is a natural candidate for $H_P$ (while, in general, $PHP$ is not!)

Counterexamples in [E.’85] and [Matolcsi-Shvidkoy’03] show, however, that it is necessary to assume that $H_P$ is \textit{densely defined}.
Zeno dynamics, continued

**Proposition:** Let $H$ be a self-adjoint operator in a separable $\mathcal{H}$, bounded from below, and let $P$ be a finite-dimensional orthogonal projection on $\mathcal{H}$. If $PH \subset Q(H)$, then for any $\psi \in \mathcal{H}$ and $t \geq 0$ we have

$$\lim_{n \to \infty} (P e^{-iHt/n} P)^n \psi = e^{-iH_P t} \psi,$$

uniformly on any compact interval of the variable $t$. 
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Without restriction on $\dim \mathcal{H}$, the formula still holds but

- convergence in a weaker topology (time averaging)
- strong convergence with added spectral projection

cf. talks by T. Ichinose and H. Neidhardt
A caricature model

An idealized description of a *quantum wire* and a family of *quantum dots*. Formally Hamiltonian acts in $L^2(\mathbb{R}^2)$ as

$$H_{\alpha,\beta} = -\Delta - \alpha \delta(x - \Sigma) + \sum_{i=1}^{n} \tilde{\beta}_i \delta(x - y^{(i)}) , \quad \alpha > 0 ,$$

where $\Sigma := \{(x_1, 0); \, x_1 \in \mathbb{R}^2\}$ and $\Pi := \{y^{(i)}\}_{i=1}^{n} \subset \mathbb{R}^2 \setminus \Sigma$.
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Singular interactions defined conventionally through b.c.: we have $\partial_{x_2} \psi(x_1, 0+) - \partial_{x_2} \psi(x_1, 0-) = -\alpha \psi(x_1, 0)$ for the line; around $y^{(i)}$ the wave functions have to behave as

$$\psi(x) = -\frac{1}{2\pi} \log |x - y^{(i)}| L_0(\psi, y^{(i)}) + L_1(\psi, y^{(i)}) + O(|x - y^{(i)}|),$$

where the generalized b.v. $L_j(\psi, y^{(i)}), \ j = 0, 1,$ satisfy

$$L_1(\psi, y^{(i)}) + 2\pi \beta_i L_0(\psi, y^{(i)}) = 0, \quad \beta_i \in \mathbb{R}$$
We introduce auxiliary Hilbert spaces, $\mathcal{H}_0 := L^2(\mathbb{R})$ and $\mathcal{H}_1 := \mathbb{C}^n$, and trace maps $\tau_j : W^{2,2}(\mathbb{R}^2) \to \mathcal{H}_j$ defined by $\tau_0 f := f \upharpoonright \Sigma$ and $\tau_1 f := f \upharpoonright \Pi$, 

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Resolvent by Krein-type formula

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canonical embeddings of free resolvent \( R(z) \) to \( \mathcal{H}_i \) by \( R_{i,L}(z) := \tau_i R(z) : L^2 \rightarrow \mathcal{H}_i \), \( R_{L,i}(z) := [R_{i,L}(z)]^* \), and \( R_{j,i}(z) := \tau_j R_{L,i}(z) : \mathcal{H}_i \rightarrow \mathcal{H}_j \), and
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\[
\Gamma_{ij}(z)g := -R_{i,j}(z)g \quad \text{for} \quad i \neq j \quad \text{and} \quad g \in \mathcal{H}_j ,
\]

\[
\Gamma_{00}(z)f := [\alpha^{-1} - R_{0,0}(z)] f \quad \text{if} \quad f \in \mathcal{H}_0 ,
\]

\[
\Gamma_{11}(z)\varphi := \left(s_\beta(z)\delta_{kl} - G_z(y^{(k)}, y^{(l)})(1 - \delta_{kl})\right) \varphi ,
\]

with \( s_\beta(z) := \beta + s(z) := \beta + \frac{1}{2\pi} \left(\ln \frac{\sqrt{z}}{2i} - \psi(1)\right) \).
Resolvent by Krein-type formula

To invert it we define the “reduced determinant”

\[ D(z) := \Gamma_{11}(z) - \Gamma_{10}(z)\Gamma_{00}(z)^{-1}\Gamma_{01}(z) : H_1 \to H_1, \]
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then an easy algebra yields expressions for “blocks” of \([\Gamma(z)]^{-1}\) in the form

\[
[\Gamma(z)]_{11}^{-1} = D(z)^{-1},
\]

\[
[\Gamma(z)]_{00}^{-1} = \Gamma_{00}(z)^{-1} + \Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1},
\]

\[
[\Gamma(z)]_{01}^{-1} = -\Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1},
\]

\[
[\Gamma(z)]_{10}^{-1} = -D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1};
\]

thus to determine singularities of \([\Gamma(z)]^{-1}\) one has to find the null space of \(D(z)\)
We can write \( R_{\alpha,\beta}(z) \equiv (H_{\alpha,\beta} - z)^{-1} \) also as a perturbation of the “line only” Hamiltonian \( \tilde{H}_\alpha \) with the resolvent

\[
R_\alpha(z) = R(z) + R_{L0}(z) \Gamma_{00}^{-1} R_{0L}(z)
\]

We define \( R_{\alpha;1L}(z) : \mathcal{H}_1 \to L^2 \) by \( R_{\alpha;1L}(z) \psi := R_\alpha(z) \psi \upharpoonright \Pi \) for \( \psi \in L^2 \) and \( R_{\alpha;1L}(z) := R^*_{\alpha;1L}(z) \). Then we have the result:
Resolvent by Krein-type formula

We can write $R_{\alpha,\beta}(z) \equiv (H_{\alpha,\beta} - z)^{-1}$ also as a perturbation of the “line only” Hamiltonian $\tilde{H}_\alpha$ with the resolvent

$$R_\alpha(z) = R(z) + R_{L0}(z)\Gamma_{00}^{-1}R_{0L}(z)$$

We define $R_{\alpha;L1}(z) : \mathcal{H}_1 \to L^2$ by $R_{\alpha;1L}(z)\psi := R_\alpha(z)\psi |_{\Pi}$ for $\psi \in L^2$ and $R_{\alpha;L1}(z) := R_{\alpha;1L}^*(z)$. Then we have the result:

**Theorem** [E.-Kondej, 2004]: For $z \in \rho(H_{\alpha,\beta})$ with $\text{Im } z > 0$ the resolvent $R_{\alpha,\beta}(z) := (H_{\alpha,\beta} - z)^{-1}$ equals

$$R_{\alpha,\beta}(z) = R(z) + \sum_{i,j=0}^{1} R_{L,i}(z)\Gamma(z)^{-1}_{ij}R_{j,L}(z)$$

$$= R_\alpha(z) + R_{\alpha;L1}(z)D(z)^{-1}R_{\alpha;1L}(z)$$
Resonance poles

The decay is due to the *tunneling between points and line*. It is absent if the interaction is “switched off” (i.e., line “put to an infinite distance”); the corresponding *free Hamiltonian* is $\tilde{H}_β := H_{0,β}$. It has $m$ eigenvalues, $1 ≤ m ≤ n$; we assume that they satisfy the condition

$$-rac{1}{4} α^2 < ε_1 < \cdots < ε_m < 0 \quad \text{and} \quad m > 1,$$

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Let us specify the interactions sites by their Cartesian coordinates, $y^{(i)} = (c_i, a_i)$. We also introduce the notations $a = (a_1, \ldots, a_n)$ and $d_{ij} = |y^{(i)} - y^{(j)}|$ for the distances in $\Pi$.

To find resonances in our model we rely on a BS-type argument; our aim is to find zeros of the function $D(\cdot)$.
Resonance poles, continued

We seek analytic continuation of $D(\cdot)$ across $(-\frac{1}{4}\alpha^2, 0) \subset \mathbb{R}$ denoting it as $D(\cdot)^{(-1)}$. The first component of $\Gamma_{11}(\cdot)^{(-1)}$ is obtained easily. To find the second one let us introduce

$$\mu_{ij}(z, t) := \frac{i\alpha}{2^5\pi} \frac{(\alpha - 2i(z - t)^{1/2}) e^{i(z-t)^{1/2}(|a_i|+|a_j|)}}{t^{1/2}(z - t)^{1/2}} e^{it^{1/2}(c_i-c_j)}.$$

Then the matrix elements of $(\Gamma_{10} \Gamma_{01})^{(-1)}(\cdot)$ are

$$\theta_{ij}^{(-1)}(\lambda) = -\int_0^\infty \frac{\mu_{ij}^0(\lambda, t)}{t - \lambda - \alpha^2/4} \, dt - 2g_{\alpha,ij}(\lambda)$$

where

$$g_{\alpha,ij}(z) := \frac{i\alpha}{(z + \alpha^2/4)^{1/2}} e^{-\alpha(|a_i|+|a_j|)/2} e^{i(z+\alpha^2/4)^{1/2}(c_i-c_j)};$$

the values at the segment and in $\mathbb{C}_+$ are expressed similarly.
Resonance poles, continued

Then we can express $\det D^{-1}(z)$. To study weak-coupling asymptotics it is useful to introduce a reparametrization

$$\tilde{b}(a) \equiv (b_1(a), \ldots, b_n(a)), \quad b_i(a) = e^{-|a_i|\sqrt{-\epsilon_i}}$$

denoting the quantity of interest as $\eta(\tilde{b}, z) = \det D^{-1}(a, z)$
Resonance poles, continued

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If $\tilde{b} = 0$ the zeros are, of course, ev’s of the point-interaction Hamiltonian $\tilde{H}_\beta$. Using implicit-function theorem we find the following weak-coupling asymptotic expansion,

$$z_i(b) = \epsilon_i + O(b) + iO(b) \quad \text{where} \quad b := \max_{1 \leq i \leq m} b_i$$
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\textit{Remark:} This model can exhibit also other long-living resonances due to \textit{weakly violated mirror symmetry}, however, we are not going to consider them here.
Dot states

By assumption there is a nontrivial discrete spectrum of $\tilde{H}_\beta$ embedded in $(-\frac{1}{4}\alpha^2, 0)$. Let us denote the corresponding normalized eigenfunctions $\psi_j$, $j = 1, \ldots, m$, given by

$$\psi_j(x) = \sum_{i=1}^{m} d_i^{(j)} \phi_i^{(j)}(x), \quad \phi_i^{(j)}(x) := \sqrt{-\epsilon_j} K_0\left(\sqrt{-\epsilon_j} |x - y^{(i)}|\right),$$

where vectors $d^{(j)} \in \mathbb{C}^m$ solve the equation $\Gamma_{11}(\epsilon_j)d^{(j)} = 0$ and the normalization condition, $\|\phi_i^{(j)}\| = 1$, reads

$$|d^{(j)}|^2 + 2\text{Re} \sum_{i=2}^{m} \sum_{k=1}^{i-1} d_i^{(j)} d_k^{(j)} (\phi_i^{(j)}, \phi_k^{(j)}) = 1.$$

In particular, if the distances in $\Pi$ are large (the natural length scale is given by $(-\epsilon_j)^{-1/2}$), the cross terms are small and each $|d^{(j)}|$ is close to one.
Decay of the dot states

Now we specify the unstable system identifying its Hilbert space $P\mathcal{H}$ with the span of $\psi_1, \ldots, \psi_m$. If it is prepared at $t = 0$ in a state $\psi \in P\mathcal{H}$, then the undisturbed decay law is

$$P_\psi(t) = \| P e^{-iH_{\alpha,\beta} t} \psi \|^2$$
Decay of the dot states

Now we specify the unstable system identifying its Hilbert space $PH$ with the span of $\psi_1, \ldots, \psi_m$. If it is prepared at $t = 0$ in a state $\psi \in PH$, then the undisturbed decay law is

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Our model is similar to (multidimensional) Friedrichs model, therefore modifying the standard argument [Demuth’76], cf. [E.-Ichinose-Kondej’05], one can check that in the weak-coupling situation the leading term in $P_\psi(t)$ will come from the appropriate semigroup evolution on $PH$, in particular, for the basis states $\psi_j$ we will have a dominantly exponential decay, $P_{\psi_j}(t) \approx e^{-\Gamma_j t}$ with $\Gamma_j = 2 \text{Im } z_j(b)$.
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Remark: The long-time behaviour of $P_{\psi_j}(t)$ is different from Friedrichs model, but this is not important here
Stable and Zeno dynamics

Suppose now that we perform the Zeno measurement at our system. We have $\dim P < \infty$ and $PH \subset Q(H_{\alpha,\beta})$, so $H_P = PH_{\alpha,\beta}P$ with the following matrix representation

$$(\psi_j, H_P \psi_k) = \delta_{jk} \epsilon_j - \alpha \int \bar{\psi}_j(x_1,0) \psi_k(x_1,0) \, dx_1,$$

where the first term corresponds, of course, to $\tilde{H}_\beta$. 
Stable and Zeno dynamics

Suppose now that we perform the Zeno measurement at our system. We have \( \dim P < \infty \) and \( P \mathcal{H} \subset \mathcal{Q}(H_{\alpha,\beta}) \), so \( H_P = P H_{\alpha,\beta} P \) with the following matrix representation

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(\psi_j, H_P \psi_k) = \delta_{jk} \epsilon_j - \alpha \int \sum \bar{\psi}_j(x_1, 0) \psi_k(x_1, 0) \, dx_1,
\]

where the first term corresponds, of course, to \( \tilde{H}_\beta \).

**Theorem** [E.-Ichinose-Kondej’05]: The two dynamics do not differ significantly for times satisfying

\[
t \ll C \epsilon^2 \sqrt{-\epsilon |\tilde{a}|},
\]

where \( C \) is a positive number and \( |\tilde{a}| = \min_i |a_i|, \epsilon = \max_i \epsilon_i \).
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$$(\psi_j, H_P \psi_k) = \delta_{jk} \epsilon_j - \alpha \int_\Sigma \tilde{\psi}_j(x_1, 0) \psi_k(x_1, 0) \, dx_1,$$

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**Theorem [E.-Ichinose-Kondej’05]:** The two dynamics do not differ significantly for times satisfying

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where $C$ is a positive number and $|\tilde{a}| = \min_i |a_i|$, $\epsilon = \max_i \epsilon_i$

**Proof:** The norm of $U_t := (e^{-i\tilde{H}_{\beta}t} - e^{-iH_P t})P$ is small as long as $t \|(\tilde{H}_{\beta} - H_P)P\| \ll 1$; to see when this is true one has to estimate contribution of the cross-terms. ☐
Now, what about anti-Zeno?

Let us now return to “Zeno-type” non-decay probability, $M_n(t) = P_{\psi}(t/n)P_{\psi_1}(t/n) \cdots P_{\psi_{n-1}}(t/n)$, where $\psi_{j+1}$ are as before, in particular, to the formula

$$M_n(t) = (P_{\psi}(t/n))^n$$

for $\dim P = 1$. 
Now, what about anti-Zeno?

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where \( \psi_{j+1} \) are as before, in particular, to the formula
\[ M_n(t) = (P_{\psi}(t/n))^n \]
for \( \dim P = 1 \). Since \( \lim_{n \to \infty} (f(t/n))^n = \exp\{\dot{f}(0+)t\} \) if \( f(0) = 1 \) and the one-sided derivative \( \dot{f}(0+) \) exists we see that \( M(t) := \lim_{n \to \infty} M_n(t) = 0 \) for \( \forall t > 0 \) if \( \dot{P}(0+) = -\infty \), and the same is true if \( \dim P > 1 \) provided the derivative \( \dot{P}_{\psi}(0+) \) has such a property for any \( \psi \in \mathcal{PH} \).
Now, what about anti-Zeno?

Let us now return to “Zeno-type” non-decay probability, $M_n(t) = P_{\psi}(t/n)P_{\psi_1}(t/n) \cdots P_{\psi_{n-1}}(t/n)$, where $\psi_{j+1}$ are as before, in particular, to the formula

$$M_n(t) = (P_{\psi}(t/n))^n$$

for $\dim P = 1$. Since $\lim_{n \to \infty} (f(t/n)^n = \exp\{-f(0+)t\}$ if $f(0) = 1$ and the one-sided derivative $f'(0+)$ exists we see that $M(t) := \lim_{n \to \infty} M_n(t) = 0$ for $\forall t > 0$ if $\dot{P}(0+) = -\infty$, and the same is true if $\dim P > 1$ provided the derivative $\dot{P}_\psi(0+)$ has such a property for any $\psi \in \mathcal{P}$. It is idealization, of course, but validity of such idealizations is the heart and soul of theoretical physics and has the same fundamental significance as the reproducibility of experimental data [Bratteli-Robinson’79].
Decay probability estimate

We need to estimate the quantity $1 - P(t)$, in other words $(\psi, P\psi) - (\psi, e^{iHt}Pe^{-iHt}\psi)$. We rewrite it as

$$1 - P(t) = 2 \text{Re} (\psi, P(I - e^{-iHt})\psi) - \| P(I - e^{-iHt})\psi \|^2$$
Decay probability estimate

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$$1 - P(t) = 2 \text{Re} (\psi, P(I - e^{-iHt})\psi) - \|P(I - e^{-iHt})\psi\|^2$$

In terms of the spectral measure $E_H$ of $H$ the r.h.s. equals

$$4 \int_{-\infty}^{\infty} \sin^2 \frac{\lambda t}{2} d\|E_H^\lambda \psi\|^2 - 4 \left\| \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin \frac{\lambda t}{2} dPE_H^\lambda \psi \right\|^2$$

By Schwarz it is non-negative; our aim is to find tighter upper and lower bounds. In particular, for $\dim P = 1$ we denote $d\omega(\lambda) := d(\psi, E_H^\lambda \psi)$ obtaining

$$4 \int_{-\infty}^{\infty} \sin^2 \frac{\lambda t}{2} d\omega(\lambda) - 4 \left| \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin \frac{\lambda t}{2} d\omega(\lambda) \right|^2$$
The one-dimensional case

Let first \( \dim P = 1 \). One can employ the spectral-measure normalization, \( \int_{-\infty}^{\infty} d\omega(\lambda) = 1 \), to rewrite the decay probability in the following way

\[
2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \sin^2 \frac{\lambda t}{2} + \sin^2 \frac{\mu t}{2} \right) d\omega(\lambda) d\omega(\mu) \\
-4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos \frac{(\lambda - \mu)t}{2} \sin \frac{\lambda t}{2} \sin \frac{\mu t}{2} d\omega(\lambda) d\omega(\mu),
\]

or equivalently

\[
1 - P(t) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2 \frac{(\lambda - \mu)t}{2} d\omega(\lambda) d\omega(\mu)
\]

We can thus try to estimate the integrated function
An estimate from above

Take $\alpha \in (0, 2]$. Using $|x|^\alpha \geq |\sin x|^\alpha \geq \sin^2 x$ together with $|\lambda - \mu|^\alpha \leq 2^\alpha (|\lambda|^\alpha + |\mu|^\alpha)$ we infer from the above formula

$$
\frac{1 - P(t)}{t^\alpha} \leq 2^{1-\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\lambda - \mu|^\alpha \, d\omega(\lambda) \, d\omega(\mu)
$$

$$
\leq 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|\lambda|^\alpha + |\mu|^\alpha) \, d\omega(\lambda) \, d\omega(\mu) \leq 4\langle |H|^\alpha \rangle_\psi
$$
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$$\frac{1 - P(t)}{t^\alpha} \leq 2^{1-\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\lambda - \mu|^\alpha d\omega(\lambda) d\omega(\mu)$$

$$\leq 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|\lambda|^\alpha + |\mu|^\alpha) d\omega(\lambda) d\omega(\mu) \leq 4 \langle |H|^\alpha \rangle_{\psi}$$

Hence $1 - P(t) = \mathcal{O}(t^\alpha)$ if $\psi \in D(|H|^\alpha/2)$. If this is true for some $\alpha > 1$ we have Zeno effect – which is a slightly weaker sufficient condition than the earlier stated one.
An estimate from above

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$$
\frac{1 - P(t)}{t^\alpha} \leq 2^{1-\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\lambda - \mu|^\alpha d\omega(\lambda) d\omega(\mu)
$$

$$
\leq 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|\lambda|^\alpha + |\mu|^\alpha) d\omega(\lambda) d\omega(\mu) \leq 4 \langle |H|^{\alpha} \rangle \psi
$$

Hence $1 - P(t) = \mathcal{O}(t^\alpha)$ if $\psi \in D(|H|^{\alpha/2})$. If this is true for some $\alpha > 1$ we have Zeno effect – which is a slightly weaker sufficient condition than the earlier stated one.

By negation, $\psi \notin D(|H|^{1/2})$ is a necessary condition for the anti-Zeno effect. Notice that in case $\psi \in \mathcal{H}_{ac}(H)$ the same follows from Lipschitz regularity, since $P(t) = |\hat{\omega}(t)|^2$ and $\hat{\omega}$ is bd and uniformly $\alpha$-Lipschitz iff $\int_{\mathbb{R}} \omega(\lambda) (1 + |\lambda|^\alpha) d\lambda < \infty$.
An estimate from below

To find a *sufficient condition* note that for $\lambda, \mu \in [-1/t, 1/t]$ there is a positive $C'$ independent of $t$ such that

$$\left| \sin \left( \frac{(\lambda - \mu)t}{2} \right) \right| \geq C'|\lambda - \mu|t;$$

one can make the constant explicit but it is not necessary.
An estimate from below

To find a *sufficient condition* note that for \( \lambda, \mu \in [-1/t, 1/t] \) there is a positive \( C \) independent of \( t \) such that

\[
\left| \sin \left( \frac{\lambda - \mu}{2} t \right) \right| \geq C |\lambda - \mu| t ;
\]

one can make the constant explicit but it is not necessary. Consequently, we have the estimate

\[
1 - P(t) \geq 2 C^2 t^2 \int_{-1/t}^{1/t} d\omega(\lambda) \int_{-1/t}^{1/t} d\omega(\mu) (\lambda - \mu)^2
\]

which in turn implies

\[
\frac{1 - P(t)}{t} \geq 4 C^2 t \left\{ \int_{-1/t}^{1/t} \lambda^2 d\omega(\lambda) \int_{-1/t}^{1/t} d\omega(\lambda) - \left( \int_{-1/t}^{1/t} \lambda d\omega(\lambda) \right)^2 \right\}
\]
Sufficient conditions

The AZ effect occurs if the r.h.s. diverges as \( t \to 0 \), e.g., if

\[
\int_{-N}^{N} \lambda^2 \, d\omega(\lambda) \int_{-N}^{N} d\omega(\lambda) - \left( \int_{-N}^{N} \lambda \, d\omega(\lambda) \right)^2 \geq cN^\alpha
\]

holds for any \( N \) and some \( c > 0, \alpha > 1 \)
Sufficient conditions

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$$

holds for any $N$ and some $c > 0$, $\alpha > 1$

We can also write it in a more compact form: introduce $H^\beta_N := H^\beta E_H(\Delta_N)$ with the spectral cut-off to the interval $\Delta_N := (-N, N)$, in particular, denote $I_N := E_H(-N, N)$.

The sufficient condition then reads

$$
\left( \langle H^2_N \rangle_\psi \langle I_N \rangle_\psi - \langle H_N \rangle_\psi^2 \right)^{-1} = o(N) \quad \text{as} \quad N \to \infty
$$
More on the one-dimensional case

*Remark:* Notice that the condition does *not* require the Hamiltonian $H$ to be below unbounded, in contrast to exponential exponential decay; it is enough that the spectral distribution has a slow decay in one direction only.
More on the one-dimensional case

**Remark:** Notice that the condition does not require the Hamiltonian $H$ to be below unbounded, in contrast to exponential exponential decay; it is enough that the spectral distribution has a slow decay in one direction only.

**Example:** Consider $H$ bd from below and $\psi$ from $\mathcal{H}_{ac}(H)$ s.t. $\omega(\lambda) \approx c\lambda^{-\beta}$ as $\lambda \to +\infty$ for some $c > 0$ and $\beta \in (1, 2)$. While $\int_{-N}^{N} \omega(\lambda) \, d\lambda \to 1$, the other two integrals diverge giving

$$cN^{3-\beta} - c^2 N^{4-2\beta}$$

as the asymptotic behavior of the l.h.s., where the first term is dominating; it gives $\dot{P}(0+) = -\infty$ so AZ effect occurs.
Remark: Notice that the condition does not require the Hamiltonian $H$ to be below unbounded, in contrast to exponential exponential decay; it is enough that the spectral distribution has a slow decay in one direction only.

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as the asymptotic behavior of the l.h.s., where the first term is dominating; it gives $\dot{P}(0+) = -\infty$ so AZ effect occurs.

Remarks: For $\beta > 2$ we have Zeno effect, so the Z-AZ gap is rather narrow! Also, $\beta = 2$ with a cut-off may give rapid oscillations around $t = 0$ obscuring existence of Zeno limit.
Multiple degrees of freedom

Let \( \dim P > 1 \) and denote by \( \{\chi_j\} \) an orthonormal basis in \( PH \). The second term in the decay-probability formula is

\[
-4 \sum_m \left| \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin \frac{\lambda t}{2} d(\chi_m, E^H \psi) \right|^2
\]
Multiple degrees of freedom

Let $\dim P > 1$ and denote by $\{\chi_j\}$ an orthonormal basis in $P\mathcal{H}$. The second term in the decay-probability formula is

$$-4 \sum_m \left| \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin \frac{\lambda t}{2} d(\chi_m, E^H_{\lambda} \psi) \right|^2$$

We also expand $\psi = \sum_j c_j \chi_j$ with $\sum_j |c_j|^2 = 1$ and denote $d\omega_{jk}(\lambda) := d(\chi_j, E^H_{\lambda} \chi_k)$, which is real-valued and symmetric w.r.t. index interchange. Using $d\|E^H_{\lambda} \psi\|^2 = \sum_{jk} \bar{c}_j c_k d\omega_{jk}(\lambda)$ we can cast the decay-probability into the form

$$1 - P(t) = 4 \sum_{jk} \bar{c}_j c_k \left\{ \int_{-\infty}^{\infty} \sin^2 \frac{\lambda t}{2} d\omega_{jk}(\lambda) \right. \left. - \sum_m \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin \frac{\lambda t}{2} d\omega_{jm}(\lambda) \int_{-\infty}^{\infty} e^{i\mu t/2} \sin \frac{\mu t}{2} d\omega_{km}(\mu) \right\}$$

(-6)
Multiple degrees of freedom, contd

If \( \dim P = \infty \) one has to check convergence of the series and correctness of interchanging of the summation and integration; it is done by means of Parseval relation.
Multiple degrees of freedom, contd

If \( \dim P = \infty \) one has to check convergence of the series and correctness of interchanging of the summation and integration; it is done by means of Parseval relation.

Next we employ normalization, \( \int_{-\infty}^{\infty} d\omega_{jk}(\lambda) = \delta_{jk} \), to derive

\[
1 - P(t) = 2 \sum_{jkm} \bar{c}_j c_k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2 \frac{(\lambda - \mu)t}{2} d\omega_{jm}(\lambda) d\omega_{km}(\mu)
\]

which can be also written concisely as

\[
1 - P(t) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2 \frac{(\lambda - \mu)t}{2} (\psi, dE^H_\lambda PdE^H_\mu \psi)
\]
General sufficient condition

Since \( \left| \sin \left( \frac{\lambda - \mu}{2} t \right) \right| \geq C |\lambda - \mu| t \) holds for \( |\mu t|, |\lambda t| < 1 \) we get

\[
1 - P(t) \geq 2C^2 t^2 \int_{-1/t}^{1/t} \int_{-1/t}^{1/t} (\lambda - \mu)^2 (\psi, dE^H_\lambda P dE^H_\mu \psi)
\]

\[
= 4C^2 t^2 \int_{-1/t}^{1/t} \int_{-1/t}^{1/t} (\lambda^2 - \lambda \mu) (\psi, dE^H_\lambda P dE^H_\mu \psi)
\]

\[
= 4C^2 t^2 \left\{ (\psi, H^2_{1/t} P I_{1/t} \psi) - \| P H_{1/t} \psi \|^2 \right\}
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\]

\[
= 4C^2 t^2 \left\{ (\psi, H^2_{1/t} PI_{1/t} \psi) - \| PH_{1/t} \psi \|^2 \right\}
\]

Let us summarize the results:

**Theorem [E.’05]:** In the above notation, suppose that

\[
\left( \langle H^2_N PI_N \rangle \psi - \| PH_N \psi \|^2 \right)^{-1} = o(N)
\]

holds as \( N \to \infty \) uniformly w.r.t. \( \psi \in \mathcal{P} \mathcal{H} \), then the permanent observation causes anti-Zeno effect.
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