



# On the discrete spectrum of soft quantum waveguides

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in Memory of Academician V.S. Vladimirov**

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# Whom we commemorate



A great mathematicians whose books influenced many people, and also a member of a generation whose life was no rosy garden walk indeed

# Geometrically induced bound states



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There is a huge number of related results involving systems in other dimensions and different geometric perturbations; for a survey and bibliography we refer to



P.E., H. Kovařík: *Quantum Waveguides*, Springer, Cham 2015

# The interest in related models



Apart from a purely mathematical interest – new solutions to one of the most studied equations! – such geometrically induced bound states are of practical importance as models of various *waveguide effects*, mainly in quantum theory, but also in *electromagnetism* or *acoustics*



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This motivated an alternative approach through '*leaky quantum wires*' which works with singular Schrödinger operators formally written as  $-\Delta - \alpha\delta(x - \Gamma)$  with  $\alpha > 0$ ,  $\Gamma$  being is a curve, a graph, or more generally, a complex of lower dimensionality, cf. Chapter 10 in [EK15, loc.cit.] and



P.E., T. Ichinose: Geometrically induced spectrum in curved leaky wires, *J. Phys. A: Math. Gen.* **34** (2001), 1439–1450.

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- $|\Gamma(s) - \Gamma(s')| \geq c|s - s'|$  holds for some  $c \in (0, 1)$
- $\Gamma$  is *asymptotically straight*: there are  $d > 0$ ,  $\mu > \frac{1}{2}$  and  $\omega \in (0, 1)$  such that

$$1 - \frac{|\Gamma(s) - \Gamma(s')|}{|s - s'|} \leq d [1 + |s + s'|^{2\mu}]^{-1/2}$$

in the sector  $S_\omega := \{(s, s') : \omega < \frac{s}{s'} < \omega^{-1}\}$

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## Theorem

Under these assumptions,  $\sigma_{\text{ess}}(-\Delta_{\delta, \alpha}) = [-\frac{1}{4}\alpha^2, \infty)$  and  $-\Delta_{\delta, \alpha}$  has at least one eigenvalue below the threshold  $-\frac{1}{4}\alpha^2$ .

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Let  $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  be an infinite and smooth planar curve without self-intersections, parametrized by its arc length  $s$ . We introduce the signed curvature  $\gamma : \gamma(s) = (\dot{\Gamma}_2 \ddot{\Gamma}_1 - \dot{\Gamma}_1 \ddot{\Gamma}_2)(s)$  and assume that

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- b  $\gamma$  is either of *compact support*,  $\text{supp } \gamma \subset [-s_0, s_0]$  for some  $s_0 > 0$ , or  $\Gamma$  is  $C^4$ -smooth and  $\gamma(s)$  together with its first and second derivatives tend to zero as  $|s| \rightarrow \infty$ ,

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- c  $|\Gamma(s) - \Gamma(s')| \rightarrow \infty$  holds as  $|s - s'| \rightarrow \infty$ .

excluding thus U-shaped curves and their various modifications

# The interaction support



We can reconstruct the curve from  $\gamma$ , up to Euclidean transformations:  
putting  $\beta(s_2, s_1) := \int_{s_1}^{s_2} \gamma(s) ds$ , we have

$$\Gamma(s) = \left( x_1 + \int_{s_0}^s \cos \beta(s_1, s_0) ds_1, x_2 - \int_{s_0}^s \sin \beta(s_1, s_0) ds_1 \right)$$

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for some  $s_0 \in \mathbb{R}$  and  $x = (x_1, x_2) \in \mathbb{R}^2$ . Next we define the strip  $\Omega^a$  by

$$\Omega^a := \{x \in \mathbb{R}^2 : \text{dist}(x, \Gamma) < a\},$$

in particular,  $\Omega_0^a := \mathbb{R} \times (-a, a)$  corresponds to a straight line for which we use the symbol  $\Gamma_0$

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- Ⓧ  $\Omega^a$  *does not intersect itself*, in particular,  $a \|\gamma\|_\infty < 1$  holds for the strip halfwidth of  $\Gamma$

which ensures that the points of  $\Omega^a$  can be uniquely parametrized as follows,

$$x(s, u) = (\Gamma_1(s) - u\dot{\Gamma}_2(s), \Gamma_2(s) + u\dot{\Gamma}_1(s)),$$

where  $N(s) = (-\dot{\Gamma}_2(s), \dot{\Gamma}_1(s))$  is the unit normal vector to  $\Gamma$  at the point  $s$ .



# The potential 'ditch'



We will deal with Schrödinger operators having an attractive potential supported in  $\Omega^a$ . To this aim, we consider

- a nonnegative  $V \in L^\infty(\mathbb{R})$  with  $\text{supp } V \subset [-a, a]$   
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It is also useful to introduce the comparison operator on  $L^2(\mathbb{R})$ ,

$$h_V = -\partial_x^2 - V(x)$$

with the domain  $H^2(\mathbb{R})$  which has in accordance with (e) a nonempty and finite discrete spectrum such that

$$\epsilon_0 := \inf \sigma_{\text{disc}}(h_V) = \inf \sigma(h_V) \in (-\|V\|_\infty, 0),$$

where the ground-state eigenvalue  $\epsilon_0$  is simple and the associated eigenfunction  $\phi_0 \in H^2(\mathbb{R})$  can be chosen strictly positive

# The essential spectrum

The spectrum of  $H_{\Gamma, \nu}$  is easily found when  $\Gamma$  is straight:

$$\sigma(H_{\Gamma_0, \nu}) = \sigma_{\text{ess}}(H_{\Gamma_0, \nu}) = [\epsilon_0, \infty)$$



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If the ditch is straight outside a compact, or at least asymptotically straight in the sense of (b), the essential spectrum is preserved:

## Proposition

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*Proof idea:* If  $\Gamma$  is straight outside a compact, the result is obtained by combination of Weyl's criterion and bracketing. In the other case, one brackets using strip neighborhoods of the 'tails' on  $\Omega^a$  and passes to the unitarily equivalent operator

$$H_{\pm}^{(j)} = h_V^N(u_1) \otimes (-\partial_s^2)_N + V_{\gamma}(s, u)$$

$$V_{\gamma}(s, u) := -\frac{\gamma(s)^2}{4(1+u\gamma(s))^2} + \frac{u\dot{\gamma}(s)}{2(1+u\gamma(s))^3} - \frac{5}{4} \frac{u^2\dot{\gamma}(s)^2}{(1+u\gamma(s))^4}$$

with the effective potential satisfying  $V_{\gamma}(s, u) \rightarrow 0$  as  $|s| \rightarrow \infty$ . □



# Asymptotic results



The 'hard-wall' and 'leaky-wire' results mentioned in the introduction provide some insight. For instance,  $-\Delta - \alpha\delta(x - \Gamma)$  can be obtained as a limit of Schrödinger operators with *suitably scaled regular potentials*,

$$V_\varepsilon : V_\varepsilon(u) = \frac{1}{\varepsilon} V\left(\frac{u}{\varepsilon}\right)$$

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Since this convergence is of *norm-resolvent type* we arrive easily at

## Proposition

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# Asymptotic results, continued



Consider now a *flat-bottom* waveguide referring to the potential

$$V_{J,0}(u) = V_0 \chi_J(u), \quad V_0 > 0,$$

where  $\chi_J$  is the indicator function of an interval  $J = [-a_1, a_2] \subset [-a_0, a_0]$

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we can easily prove the following result:

## Proposition

*Suppose that  $\Gamma$  is not straight and assumptions (a)–(d) are satisfied, then the operator  $H_{\Gamma, V_{J,0}}$  referring to the flat-bottom potential has nonempty discrete spectrum for all  $V_0$  large enough*

# Birman-Schwinger analysis



This will be our main tool. Given a function  $V$  and  $z \in \mathbb{C} \setminus \mathbb{R}_+$  we put

$$K_{\Gamma, V}(z) := \tilde{V}^{1/2}(-\Delta - z)^{-1} \tilde{V}^{1/2}$$

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## Proposition

$z \in \sigma_{\text{disc}}(H_{\Gamma, V})$  holds if and only if  $1 \in \sigma_{\text{disc}}(K_{\Gamma, V}(z))$ . The function  $\kappa \mapsto K_{\Gamma, V}(-\kappa^2)$  is continuous and decreasing in  $(0, \infty)$ , tending to zero in the norm topology, that is,  $\|K_{\Gamma, V}(-\kappa^2)\| \rightarrow 0$  holds as  $\kappa \rightarrow \infty$

# Birman-Schwinger analysis, continued



Note that if  $g$  is an eigenfunction of  $K_{\Gamma, V}(-\kappa^2)$  with eigenvalue one, the corresponding eigenfunction of  $H_{\Gamma, V}$  is given by

$$\phi(x) = \int_{\text{supp } \tilde{V}} G_{\kappa}(x, x') \tilde{V}(x')^{1/2} g(x') dx',$$

where  $G_{\kappa}$  is the integral kernel of  $(-\Delta + \kappa^2)^{-1}$ .

# Birman-Schwinger analysis, continued



Note that if  $g$  is an eigenfunction of  $K_{\Gamma, V}(-\kappa^2)$  with eigenvalue one, the corresponding eigenfunction of  $H_{\Gamma, V}$  is given by

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Using the knowledge of the Laplacian resolvent we can write the action of  $K_{\Gamma, V}(-\kappa^2)$  explicitly: it is an integral operator with the kernel

$$K_{\Gamma, V}(x, x'; -\kappa^2) = \frac{1}{2\pi} \tilde{V}^{1/2}(x) K_0(\kappa|x - x'|) \tilde{V}^{1/2}(x'),$$

where  $K_0$  is the Macdonald function, mapping  $L^2(\Omega^a)$  to itself.

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In analogy with [E-Ichinose'01, loc.cit.] the idea is to treat the geometry of  $\Omega^a$  as a perturbation of the straight case

## Straightening the strip



First we 'straighten' strip as one does it for the 'hard-wall' waveguides.

Passing from the Cartesian coordinates to  $s, u$  amounts to a unitary map

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The operator  $K_{\Gamma, V}(-\kappa^2)$  transforms to the unitarily equivalent one,  $\mathcal{R}_{\Gamma, V}^\kappa := UK_{\Gamma, V}(-\kappa^2)U^{-1}$ , which is an integral operator on  $L^2(\Omega_0^a)$  with the kernel

$$\mathcal{R}_{\Gamma, V}^\kappa(s, u; s', u') = \frac{1}{2\pi} W(s, u)^{1/2} K_0(\kappa|x - x'|) W(s', u')^{1/2},$$

where  $x = x(s, u)$ ,  $x' = x(s', u')$ , and the modified potential is

$$W(s, u) := (1 + u\gamma(s)) V(u)$$

# The straight case

For the straight potential ditch we have

$$\mathcal{R}_{\Gamma_0, V}^{\kappa}(s, u; s', u') = \frac{1}{2\pi} V(u)^{1/2} K_0(\kappa |x_0 - x'_0|) V(u')^{1/2},$$

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where  $|x_0 - x'_0| = [(s - s')^2 + (u - u')^2]^{1/2}$ . In the  $s$  variable the operator is of convolution type, thus we have

$$(F \otimes I) \mathcal{R}_{\Gamma_0, V}^{\kappa} (F \otimes I)^{-1} = \int_{\mathbb{R}}^{\oplus} \mathcal{R}_{\Gamma_0, V}^{\kappa}(p) dp,$$

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By assumption,  $\epsilon_0 = \inf \sigma(h_V)$ , and consequently,  $-\kappa^2 = \epsilon_0 + p^2$  belongs to the spectrum of  $H_{\Gamma_0, V}$  for any  $p \in \mathbb{R}$  as we know already

## The straight case, continued



At the same time, the operator  $\mathcal{R}_{\Gamma_0, \mathcal{V}}^{\kappa_0}$  satisfies

$$\sup \sigma(\mathcal{R}_{\Gamma_0, \mathcal{V}}^{\kappa_0}) = 1,$$

where  $\kappa_0 = \sqrt{-\epsilon_0}$ , because otherwise there would be a  $\tilde{\kappa} > \kappa_0$  such that  $1 \in \sigma(\mathcal{R}_{\Gamma_0, \mathcal{V}}^{\tilde{\kappa}})$ , and consequently,  $-\tilde{\kappa}^2 \in \sigma(H_{\Gamma_0, \mathcal{V}})$ , however, this would contradict to the already established fact that  $\sigma(H_{\Gamma_0, \mathcal{V}}) = [\epsilon_0, \infty)$

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One can relate the eigenfunction  $\phi_0$  of  $h_V$  to the eigenfunction  $g_0$  of  $\mathcal{R}_{\Gamma_0, V}^{\kappa_0}(0)$  corresponding to the unit eigenvalue. On the one hand, we have

$$g_0(u) = V^{1/2}(u)\phi_0(u),$$

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on the other hand, one can write the generalized eigenfunction associated with  $\inf \sigma(H_{\Gamma_0, V})$  as

$$f_0(s, u) = \phi_0(u) = \int_{-a}^a \frac{e^{-\kappa_0|u-u'|}}{2\kappa_0} V(u')^{1/2} g_0(u') du'$$

# Existence of bound states



## Theorem

Let assumptions (a)–(e) be valid and set

$$\begin{aligned} C_{\Gamma, V}^{\kappa}(s, u; s', u') &= \frac{1}{2\pi} \phi_0(u) V(u) [(1 + u\gamma(s)) K_0(\kappa|x(s, u) - x(s', u')|) (1 + u'\gamma(s')) \\ &\quad - K_0(\kappa|x_0(s, u) - x_0(s', u')|)] V(u') \phi_0(u') \end{aligned}$$

for all  $(s, u), (s', u') \in \Omega_0^a$ , then we have  $\sigma_{\text{disc}}(H_{\Gamma, V}) \neq \emptyset$  provided

$$\int_{\mathbb{R}^2} ds ds' \int_{-a}^a \int_{-a}^a du du' C_{\Gamma, V}^{\kappa_0}(s, u; s', u') > 0$$

holds for  $\kappa_0 = \sqrt{-\epsilon_0}$ .



P.E.: Spectral properties of soft quantum waveguides, *J. Phys. A: Math. Theor.* **53** (2020), 355302.

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In contrast to the asymptotic results above, this one has a quantitative character



# Bound state existence, proof sketch



The idea is to treat the geometry of the system, translated into the coefficients of the operator, as a perturbation of the straight case. As the essential spectrum is preserved, it is enough to find  $\psi_\eta \in L^2(\Omega_0^a)$  such that

$$(\psi, \mathcal{R}_{\Gamma, V}^{\kappa_0} \psi) - \|\psi\|^2 > 0. \quad (1)$$

A trial function combines the generalized eigenfunction, associated with the edge of the spectrum, with a mollifier which makes it an element of the Hilbert space

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A trial function combines the generalized eigenfunction, associated with the edge of the spectrum, with a mollifier which makes it an element of the Hilbert space. Inspect first the effect of the mollifier for  $\Gamma = \Gamma_0$ :

## Lemma

Let  $\psi_\eta \in L^2(\Omega_0^a)$  be of the form  $\psi_\eta(s, u) = h_\eta(s)g_0(u)$  with  $h_\eta(s) = h(\eta s)$ , where  $h \in C_0^\infty(\mathbb{R})$  and  $h(s) = 1$  in the vicinity of  $s = 0$ . Then

$$(\psi_\eta, \mathcal{R}_{\Gamma_0, \nu}^{\kappa_0} \psi_\eta) - \|\psi_\eta\|^2 = \mathcal{O}(\eta) \quad \text{as } \eta \rightarrow 0.$$

# Bound state existence, proof sketch



We can rewrite the expression to be estimated into the form

$$\int_{-a}^a \int_{-a}^a g_0(u) V(u)^{1/2} \left[ \int_{\mathbb{R}} |\hat{h}_\eta(p)|^2 \frac{e^{-\sqrt{\kappa_0^2 + p^2}|u-u'|}}{2\sqrt{\kappa_0^2 + p^2}} dp - \|h_\eta\|^2 \frac{e^{-\kappa_0|u-u'|}}{2\kappa_0} \right] \\ \times V(u')^{1/2} g_0(u') du du',$$

and it is enough to check that the square bracket is  $\mathcal{O}(\eta)$  as  $\eta \rightarrow 0$ .

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and it is enough to check that the square bracket is  $\mathcal{O}(\eta)$  as  $\eta \rightarrow 0$ .

We have  $\hat{h}_\eta(p) = \frac{1}{\eta} \hat{h}\left(\frac{p}{\eta}\right)$ , which allows us to rewrite the first term as

$$\frac{1}{\eta} \int_{\mathbb{R}} |\hat{h}(\zeta)|^2 \frac{e^{-\sqrt{\kappa_0^2 + \eta^2 \zeta^2}|u-u'|}}{2\sqrt{\kappa_0^2 + \eta^2 \zeta^2}} d\zeta = \frac{1}{\eta} \left( \frac{e^{-\kappa_0|u-u'|}}{2\kappa_0} + \mathcal{O}(\eta^2) \right),$$

and using further the relation  $\|h_\eta\|^2 = \frac{1}{\eta} \|h\|^2$  we prove the lemma.

# Bound state existence, proof concluded



Consider now the difference of the Birman-Schwinger operators

$$\mathcal{D}_{\Gamma, V}^{\kappa} := \mathcal{R}_{\Gamma, V}^{\kappa} - \mathcal{R}_{\Gamma_0, V}^{\kappa}$$

which is an integral operator with the kernel

$$\begin{aligned} \mathcal{D}_{\Gamma, V}^{\kappa}(s, u; s', u') &= \frac{1}{2\pi} \left( W(s, u)^{1/2} K_0(\kappa |x(s, u) - x(s', u')|) W(s', u')^{1/2} \right. \\ &\quad \left. - V(u)^{1/2} K_0(\kappa |x_0(s, u) - x_0(s', u')|) V(u')^{1/2} \right) \end{aligned}$$

By BS principle, a bound state existence requires  $\sup \sigma(\mathcal{R}_{\Gamma_0, V}^{\kappa_0}) > 1$ , and that happens if

$$\lim_{\eta \rightarrow 0} (\psi_{\eta}, \mathcal{D}_{\Gamma, V}^{\kappa_0} \psi_{\eta}) > 0.$$

Using the choice of the function  $h_{\eta}$ , this is equivalent to

$$\int_{\mathbb{R}^2} ds ds' \int_{-a}^a \int_{-a}^a du du' g_0(u) \mathcal{D}_{\Gamma, V}^{\kappa_0}(s, u; s', u') g_0(u') > 0,$$

which is nothing else than the condition stated in the theorem. □

# Distances involved



To use the theorem one has to compare point distances in the straight strip,

$$|x_0(s, u) - x_0(s', u')| = [(s - s')^2 + (u - u')^2]^{1/2}$$

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with those is the curved one,

$$\begin{aligned} & |x(s, u) - x(s', u')|^2 \\ &= |\Gamma(s) - \Gamma(s')|^2 + u^2 + u'^2 - 2uu' \cos \beta(s, s') + 2(u \cos \beta(s, s') - u') \int_{s'}^s \sin \beta(\xi, s') d\xi, \end{aligned}$$

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This property was decisive in the leaky wire case, [E-Ichinose'01, *loc.cit.*].

# One more existence result



## Proposition

Let  $\mathcal{V}_{\epsilon_0}$  be the family of potentials  $V$  satisfying assumptions (d), (e), and  $\inf \sigma(h_V) \leq \epsilon_0$ . Then to any  $\epsilon_0 > 0$  there exists an  $a_0 = a_0(\epsilon_0)$  such that  $\sigma_{\text{disc}}(H_{\Gamma, V}) \neq \emptyset$  holds for all  $V \in \mathcal{V}_{\epsilon_0}$  with  $\text{supp } V \subset [-a_0, a_0]$ .

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It is sufficient to consider  $\inf \sigma(h_V) = \epsilon_0$  since the family  $\{h_{\lambda V} : \lambda > 0\}$  is monotonous with the same essential spectrum.  $\sigma_{\text{disc}}(H_{\Gamma, V}) \neq \emptyset$  hold if

$$\frac{1}{2\pi} \int_{-a}^a \int_{-a}^a \phi_0(u) V(u) F(u, u') V(u') \phi_0(u) \, du du' > 0,$$

where

$$F(u, u') := \int_{\mathbb{R}^2} [(1 + u\gamma(s)) K_0(\kappa_0|x(s, u) - x(s', u')|) (1 + u'\gamma(s')) - K_0(\kappa_0|x_0(s, u) - x_0(s', u')|)] \, ds ds'$$

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The function  $F(\cdot, \cdot)$  is well defined, continuous, and we have

$$F(0, 0) = \int_{\mathbb{R}^2} [K_0(\kappa_0|\Gamma(s) - \Gamma(s')|) - K_0(\kappa_0|s - s'|)] ds ds' > 0.$$

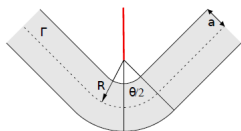
By continuity there is a neighborhood  $(-a_0, a_0) \times (-a_0, a_0)$  of  $(0, 0)$  on which  $F(u, u')$  is positive, and that in combination with the positivity of  $\phi_0 V$  concludes the proof. □

# Comments



Birman-Schwinger principle is not the only tool available; a natural alternative is to employ a *variational method*

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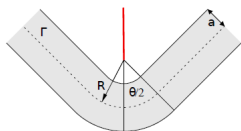


Source: the cited paper



S. Kondej, D. Krejčířík, J. Kříž: Soft quantum waveguides with a explicit cut locus, [arXiv:2007.10946](https://arxiv.org/abs/2007.10946)

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This is not the end of the story, many questions remain open, for instance

- The existence of bound states in *polygonal channels*. There is such a result obtained variationally for *crossed channels* but this fact alone does not allow us to make conclusions about a single broken channel.



S. Egger, J. Kerner, K. Pankrashkin: Bound states of a pair of particles on the half-line with a general interaction potential, *J. Spect. Theory*, to appear; [arXiv:1812.06500](https://arxiv.org/abs/1812.06500)

# More problems



- Tubular potential channels *in three dimensions*. One expects the validity of asymptotic results similar to those discussed above. If the channel profile lacks the rotational symmetry with respect to its axis  $\Gamma$ , one expects additional effects coming from the channel *torsion* giving rise repulsion in analogy with



T. Ekholm, H. Kovařík, D. Krejčířík: A Hardy inequality in twisted waveguides, *Arch. Rat. Mech. Anal.* **188**(2008), 245–264.



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- *Local perturbations* of potential channels, coming from variation either of their depth or width. This is easy if such a perturbation is 'sign-definite', in general it may be harder.

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T. Ekholm, H. Kovařík, D. Krejčířík: A Hardy inequality in twisted waveguides, *Arch. Rat. Mech. Anal.* **188**(2008), 245–264.

- *Local perturbations* of potential channels, coming from variation either of their depth or width. This is easy if such a perturbation is 'sign-definite', in general it may be harder.
- Potential channels of a more complicated geometry, in first place *branched ones* built over a metric graph. Of course, to have the problem well defined one must specify the potential in the vicinity of the graph vertices because the spectrum would depend on it.

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- Potential channels of a more complicated geometry, in first place *branched ones* built over a metric graph. Of course, to have the problem well defined one must specify the potential in the vicinity of the graph vertices because the spectrum would depend on it.
- One can ask about the *number of eigenvalues* and their properties in dependence on the system geometry. Of particular interest are the *weakly bound states* corresponding to mild geometric perturbations.

# More problems



- Another question concerns *scattering* in a bent or locally perturbed potential channel including possible *resonance effects* in narrow and sufficiently deep channels.



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- For layers the spectrum may depend on the *global* geometry of the interaction support. An example of a *conical* potential layer was found recently, properties of more general layers are of interest.



S. Egger, J. Kerner, K. Pankrashkin: Discrete spectrum of Schrödinger operators with potentials concentrated near conical surfaces, *Lett. Math. Phys.* **110** (2020), 945–968.

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- Another question concerns the influence of *external fields*. In a two-dimensional hard-wall tube we have a Hardy-type inequality that prevents the existence of weakly bound states; the question is whether this extends to soft waveguides



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- A completely new area opens when we consider a system of *many particles* interacting mutually, for instance, due to the charges they carry, confined in a soft waveguide.

# One more problem



Another question one may pose concerns the *spectral optimization* in analogy with what is known in Dirichlet and  $\delta$  potential cases



P.E., E.M. Harrell, M. Loss: Optimal eigenvalues for some Laplacians and Schrödinger operators depending on curvature, in *Mathematical Results in Quantum Mechanics*, Birkhäuser, Basel 1999; pp. 47–53.



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We consider operators  $H_{\gamma, \mu}$  corresponding the measure-type interaction

$$\mu(M) := \int_0^L \int_{-d_-}^{d_+} \chi_M(\Gamma(s) + u\nu(s)) (1 + u\gamma(s)) d\mu_{\perp}(t) ds,$$

where the positive transverse measure  $\mu_{\perp}$  describes either a regular attractive potential channel,  $\mu_{\perp}(u) = V(u)du$ , or a  $\delta$  potential.

# Shape optimization



We define  $H_{\Gamma, \mu}$  as the self-adjoint operator associated with the form

$$h_{\Gamma, \mu}[\psi] := \|\nabla \psi\|^2 - \int_{\mathbb{R}^2} |\psi|^2 d\mu, \quad \text{dom } h_{\Gamma, \mu} = H^1(\mathbb{R}^2).$$

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It is not difficult to check that the essential spectrum of  $H_{\Gamma,\mu}$  is  $[0, \infty)$  and  $\sigma_{\text{disc}}(H_{\Gamma,\mu}) \neq \emptyset$ . Let  $\mathcal{C}$  be a *circle of radius*  $\frac{L}{2\pi}$ . By  $\mu_{\circ}$  we denote the corresponding measure generated by  $\mu_{\perp}$  and giving rise to operator  $H_{\Gamma,\mu_{\circ}}$ .

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## Theorem

*The lowest eigenvalues  $\lambda_1(\mu)$  and  $\lambda_1(\mu_o)$ , respectively, of  $H_{\Gamma, \mu}$  and of  $H_{\Gamma, \mu_o}$  satisfy the inequality*

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We *conjecture* that the inequality is strict unless  $\Gamma$  and  $\mathcal{C}$  are congruent.

## Another optimization result



The claim follows by a simple *variational argument*: the appropriate trial function is obtained using the lowest eigenfunction of  $H_{\Gamma, \mu_0}$  and '*transplanting*' it to the parallel coordinates.



P.E., V. Lotoreichik: Optimization of the lowest eigenvalue of a soft quantum ring, arXiv:2011.02257 [math-ph]

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This is again easy to prove variationally; one has to check that the function  $\mathcal{J} \ni t \mapsto \|\psi|_u\|^2$  is continuous so that it attains its maximum value at some  $t_{\star} = t_{\star}(\mu) \in \mathcal{J}$ .

# It remains to say



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**Спасибо за внимание!**