On loops, cones, and stars: striving for the optimal shape

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Prague

Motto: Ubi materia, ibi geometria (Johannes Keppler)

A talk at the conference Results in Contemporary Mathematical Physics

Santiago de Chile, December 20, 2018

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Or a different question: do we have a colleague ready to examine critically every principle, even if comes from highest authorities?

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Stanislaw Łem: The Magellanic cloud

An important example

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MARK S. ASHBAUGH AND RAFAEL D. BENGURIA

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0273-0979/91 $1.00 + $.25 per page

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It is useful to go through the history of attempts to demonstrate the PPW conjecture to realize what a *tour de force* this result is

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As the first example, consider the optimization problem for a *Robin Laplacian* associated with the quadratic form

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on $H^1(\Omega)$. As long as $\alpha > 0$ the result is similar to Faber-Krahn: the principal eigenvalue $\lambda_1^\alpha(\Omega)$ is *uniquely minimized* among the sets of *the same volume* by $\lambda_1^\alpha(\mathcal{B})$ where $\mathcal{B}$ is the *ball*

Attractive Robin boundary

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Back to the Dirichlet case

Even when the boundary is Dirichlet, the situation is not simple and *topology may play role*.

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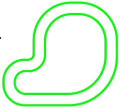
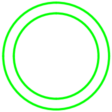
ground state of  $<$ ground state of 

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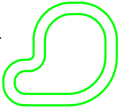
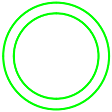
ground state of  $<$ ground state of 

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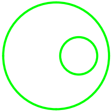
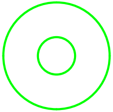
Similarly, for a *circular obstacle in circular cavity* we have[7]

ground state of  $<$ ground state of 

whenever the obstacle is off center; the minimum is reached when it is touching the boundary

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The δ-interaction supported by a manifold

A natural tool to define the corresponding singular Schrödinger operator is to employ the appropriate quadratic form, namely

$$q_{\delta,\alpha}[\psi] := \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)} - \alpha\|f|_\Gamma\|^2_{L^2(\Gamma)}$$

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If Γ is a *smooth manifold* with $\operatorname{codim}\Gamma = 1$ one can easily check that the form defines a unique self-adjoint operator $H_{\alpha,\Gamma}$, which can alternatively characterized by boundary conditions: it acts as $-\Delta$ on functions from $H^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

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This explains the formal expression as describing the attractive δ-interaction of strength $\alpha(x)$ perpendicular to Γ at the point x.

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$$\frac{\partial\psi}{\partial n}(x)\bigg|_+ - \frac{\partial\psi}{\partial n}(x)\bigg|_- = -\alpha(x)\psi(x)$$

This explains the formal expression as describing the attractive δ-interaction of strength $\alpha(x)$ perpendicular to Γ at the point x.

Alternatively, one sometimes uses the symbol $-\Delta_{\delta,\alpha}$ for this operator.

Planar loops

Let Γ be a *loop* in $\mathbb{R}^d$, $d \geq 2$, parametrized by its arc length, i.e. a piecewise differentiable function $\Gamma : [0, L] \to \mathbb{R}^d$ such that $\Gamma(0) = \Gamma(L)$ and $|\dot{\Gamma}(s)| = 1$ for all but finitely many $s \in [0, L]$

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Theorem ([E-Harrell-Loss'06])

Let $d = 2$. For any $\alpha > 0$ and $L > 0$ we have $\lambda_1(\alpha, \Gamma) \leq \lambda_1(\alpha, \mathcal{C})$, where $\mathcal{C}$ is a circle of perimeter L, the inequality being sharp unless Γ is congruent with $\mathcal{C}$.

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Proof idea: One employs a generalized *Birman-Schwinger principle* by which there is one-to-one correspondence between eigenvalues $-\kappa^2$ of $H_{\alpha,\Gamma}$ and solutions to the integral-operator equation

$$\mathcal{R}^{\kappa}_{\alpha,\Gamma}\phi = \phi\,, \quad \text{where } \mathcal{R}^{\kappa}_{\alpha,\Gamma}(s, s') := \frac{\alpha}{2\pi} K_0(\kappa|\Gamma(s) - \Gamma(s')|)$$

on $L^2([0, L])$, where K_0 is the Macdonald function

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Proof idea, continued

We employ *inequalities on mean values of chords* denoted as $C_L^p(u)$:

$$\int_0^L |\Gamma(s+u)-\Gamma(s)|^p \mathrm{d}s \le \frac{L^{1+p}}{\pi^p}\, sin^p \frac{\pi u}{L}\,, \quad p>0,\ u\in(0,\tfrac12 L]$$

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Using a variational argument together with the fact that $K_0(\cdot)$ appearing in the resolvent kernel is *strictly monotonous and convex* the optimization problem for $\mathcal{R}^\kappa_{\alpha,\Gamma}$ is reduced to the inequality $C_L^1(u)$ being thus proved □

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Remark: The (reverse) inequalities hold also for $p\in[-2,0)$ showing, e.g., that a *charged loop in the absence of gravity takes a circular form*

A discrete analogue: polymer loops

Consider the same loop as above with *point interactions* placed at the arc distances $\frac{jL}{N}$, $j = 0, \dots, N_1$, in other words, the formal Hamiltonian

$$H^N_{\alpha,\Gamma} = -\Delta + \tilde{\alpha} \sum_{j=0}^{N-1} \delta\left(x - \Gamma\left(\frac{jL}{N}\right)\right)$$

in $L^2(\mathbb{R}^d),\ d = 2, 3$, where the last term has to be properly defined

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Introduce the *generalized boundary values* as the coefficients in the expansion of H_Y^* where H_Y is the Laplacian restricted to functions vanishing at the vicinity of the points of Y

Point interactions 'necklaces'

A reminder: fixing the points $y_j \in Y$ the said expansion look as follows

$$\psi(x) = -\frac{1}{2\pi} \log|x - y_j|\, L_0(\psi, y_j) + L_1(\psi, y_j) + \mathcal{O}(|x - y_j|), \quad d = 2,$$

$$\psi(x) = \frac{1}{4\pi|x - y_j|}\, L_0(\psi, y_j) + L_1(\psi, y_j) + \mathcal{O}(|x - y_j|), \quad d = 3.$$

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Local self-adjoint extension are then given by

$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0\,, \quad \alpha \in \mathbb{R};$$

for details we refer to [AGHH'88, 05]

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Theorem ([E'06])

The ground state of $H^N_{\alpha,\Gamma}$ is uniquely maximized by a N-regular polygon

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New effects in three dimensions

In three dimensions the discrete spectrum of $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)$ *may be empty* is α is small enough. As an example, for Γ being a *sphere of radius R* bound states are known[10] to exist *iff* $\alpha R > 1$

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Theorem ([E-Fraas'09])

Let Γ_ϵ by a deformation of the sphere expressed in spherical coordinates as $r(\theta,\phi) = R(1+\epsilon\rho(\theta,\phi))$ where ρ is nonzero function of zero mean. If H_{α,Γ_0} is critical, $\sigma_{\rm disc}(H_{\alpha,\Gamma_\epsilon}) \neq \emptyset$ holds for all nonzero ϵ small enough.

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Remarks: (a) The results *fails to hold globally*: if a *surface-preserving* deformation of a critical surface is *elongated enough*, the discrete spectrum is *empty*.

(b) In contrast, deformation of a critical surface *always produces a nonvoid discrete spectrum* if it is *capacity preserving*

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More singular interactions in 2D

So far I spoke of an old stuff. Let us now look at some fresh results. Consider again a planar loop a replace δ by δ' *interaction*.

[12] V. Lotoreichik: Spectral isoperimetric inequality for the δ' interaction on a contour, arXiv:1810.05457

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So far I spoke of an old stuff. Let us now look at some fresh results. Consider again a planar loop a replace δ by *δ' interaction*. The latter can be defined by boundary condition or using the quadratic form,

$$q_{\delta',\beta}[\psi] := \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{\beta}\|[f]_\Gamma\|^2_{L^2(\Gamma)}$$

defined on $H^1(\mathbb{R}^2 \setminus \Gamma)$, where $[f]_\Gamma := f_+|_\Gamma - f_+|_\Gamma$.

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For any $\beta > 0$ we have $\max_{|\Gamma|=L} \lambda_1^\beta(\Gamma) = \lambda_1^\beta(\mathcal{C})$, where $\mathcal{C}$ is a circle of perimeter $L > 0$ and the maximum is taken over all C^2 smooth loops.

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The Birman-Schwinger method does not work in this case, one has to use instead *locally orthogonal coordinates* in a way similar to those employed in [Krejčiřík-Lotoreichik'18] to treat exterior of a Robin obstacle

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Cones

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We start with some *definitions*: Let $\mathcal{T} \subset \mathbb{S}^2$ be a C^2-smooth loop on the 2D unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ of length $|\mathcal{T}|$ without self-intersections. We distinguish between *circular* and *non-circular loops*. A circle $\mathcal{C}$ on $\mathbb{S}^2$ has, of course, the length $|\mathcal{C}| \leq 2\pi$.

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The C^2-smooth *cone* $\Sigma_R(\mathcal{T}) \subset \mathbb{R}^3$ of radius $R \in (0, \infty]$ with a C^2-smooth loop $\mathcal{T} \subset \mathbb{S}^2$ as its *cross-section* is

$$\Sigma_R(\mathcal{T}) := \left\{ r\mathcal{T} \in \mathbb{R}^3 \colon r \in [0, R) \right\};$$

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The cone $\Sigma_R(\mathcal{T})$ is called *circular* if its cross-section $\mathcal{T}$ is a circle and *non-circular* otherwise. An infinite circular cone with the cross-section length 2π is, in fact, a plane.

Results in the finite case

If $R < \infty$ it is not difficult to check that $\sigma_{\rm ess}(H_{\alpha,\Gamma}) = [0,\infty)$; we are interested in the principal eigenvalue $\lambda_1(H_{\alpha,\Gamma})$. We have[13]

Theorem ([E-Lotoreichik'17])

Let $\mathcal{C} \subset \mathbb{S}^2$ be a circle and $\mathcal{T} \subset \mathbb{S}^2$ be a C^2-smooth non-circular loop such that $L := |\mathcal{C}| = |\mathcal{T}| \in (0, 2\pi]$. Let $\Gamma_R := \Sigma_R(\mathcal{C})$ and $\Lambda_R := \Sigma_R(\mathcal{T})$ be finite cones of radius $R > 0$ with the cross-sections $\mathcal{C}$ and $\mathcal{T}$, respectively; then

- $\#\sigma_{\rm disc}(H_{\alpha,\Gamma_R}) \geq 1$ *if and only if* $\alpha > \alpha_{\rm crit}$ *for a certain value* $\alpha_{\rm crit} = \alpha_{\rm crit}(L,R) > 0$.

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- $\#\sigma_{\rm disc}(H_{\alpha,\Gamma_R}) \geq 1$ *if and only if* $\alpha > \alpha_{\rm crit}$ *for a certain value* $\alpha_{\rm crit} = \alpha_{\rm crit}(L, R) > 0$.
- $\#\sigma_{\rm disc}(H_{\alpha,\Lambda_R}) \geq 1$ *for all* $\alpha \geq \alpha_{\rm crit}$ *(the borderline case $\alpha = \alpha_{\rm crit}$ is included) and the spectral isoperimetric inequality*

$$\lambda_1(H_{\alpha,\Lambda_R}) < \lambda_1(H_{\alpha,\Gamma_R})$$

is satisfied for all $\alpha > \alpha_{\rm crit}$.

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Cones, continued

We see the effect we have encountered before with spheres:

Corollary

Any (fixed-radius, smooth, conical) deformation of a critical circular cone gives rise to a non-void discrete spectrum of the corresponding $H_{\alpha,\Gamma}$

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Moreover, we even know its *accumulation rate*: for circular cones we have[14]

$$\mathcal{N}_{-\frac{1}{4}\alpha^2 - E}(-\Delta_{\delta,\alpha}) \sim \frac{\cot\theta}{4\pi}\,|\ln E|\,, \quad E \to 0+\,,$$

and as a similar results also holds in the non-circular case[15]

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- $\#\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma_\infty}) \cap (-\infty, -\frac{1}{4}\alpha^2) \geq 1$

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Let $\mathcal{C} \subset \mathbb{S}^2$ *be a circle and* $\mathcal{T} \subset \mathbb{S}^2$ *be a* C^2*-smooth non-circular loop such that* $L := |\mathcal{C}| = |\mathcal{T}| \in (0, 2\pi)$*. Let* $\Gamma_\infty := \Sigma_\infty(\mathcal{C})$ *and* $\Lambda_\infty := \Sigma_\infty(\mathcal{T})$ *be infinite cones with the cross-sections* $\mathcal{C}$ *and* $\mathcal{T}$*, respectively; then for any* $\alpha > 0$ *we have*

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Proof sketch in the finite case

The strategy is to employ the *generalized Birman-Schwinger principle* in combination with a minimization result about the energy of knots, cf. [E-Harrell-Loss'06] and an earlier paper by Abrams et al.[16]

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To proceed, we need a suitable *parametrization of the cone*. We begin with arc-length parametrization of the cross section, $\tau\colon [0,L] \to \mathbb{S}^2$ with $|\dot\tau| \equiv 1$ and put

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Let $\mathcal{C} \subset \mathbb{S}^2$ be a circle and $\Gamma_R := \Sigma_R(\mathcal{C})$. Then the eigenfunction corresponding to the largest eigenvalue of the BS operator $S_{\Gamma_R}(\kappa)$ is rotationally invariant, i.e. it depends on the distance from the tip of the cone only.

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Now we employ an inequality related to $C_L^p(u)$ used earlier, known also from other sources:[17]

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$$\Phi_f[\mathcal{C}] < \Phi_f[\mathcal{T}].$$

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In particular, the above proposition holds with the function

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we have to exclude situations where $r=0$, $r'=0$ or $r=r'$, but this is a zero measure set. With a bit of work, this yields finally the result. $\square$

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Given $\alpha > 0$, we ask again about the *spectral threshold* of the operator H_{α,Σ_N} corresponding to the formal expression $-\Delta - \alpha\delta(x - \Sigma_n)$

An illustration

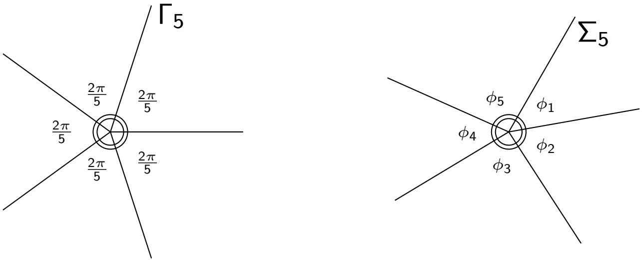

Figure: The star graphs Γ_5 and Σ_5 with $N = 5$ and $L < \infty$.

Star optimization

It is easy to see that $\sigma_{\rm ess}(H_{\alpha,\Sigma_N}) = [0,\infty)$ if $L<\infty$ and with the set $\sigma_{\rm ess}(H_{\alpha,\Sigma_N}) = [-\frac14\alpha^2,\infty)$ if $L=\infty$.

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Theorem ([E-Lotoreichik'18])

For any $\alpha>0$ we have the relation

$$\max_{\Sigma_N(L)} \lambda_1^\alpha(\Sigma_N(L)) = \lambda_1^\alpha(\Gamma_N(L)),$$

where the maximum is taken over all star graphs with $N\geq 2$ edges of a given length $L\in(0,\infty]$. If $L<\infty$ the equality is achieved iff Σ_N and Γ_N are congruent.

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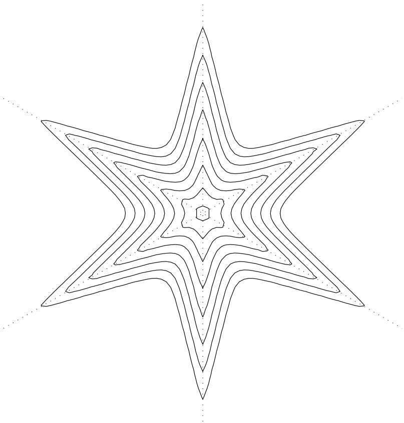
Moreover, the ground state has then some *esthetic quality*:



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To establish the relation for $L = \infty$ one uses the *strong resolvent convergence* which gives, in particular,

$$\lim_{L\to\infty} \lambda_1^\alpha(\Sigma_N(L)) = \lambda_1^\alpha(\Sigma_N(\infty))$$

and the analogous relation for symmetric stars

Stars in three dimensions

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Without being too technical, one takes the Laplacian defined on functions that are H^2 outside Γ and imposed the *generalized boundary conditions*[20] defining 2D point interaction in the cross planes to the edges of Γ

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Recall some related problems

Optimization problem for 3D stars is no doubt nontrivial. The first analogue coming to mind is the *Thomson problem*[21] about distribution of N point charges on the surface of a sphere

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Recall that a rigorous solution is known for a few small N cases, for instance, a (computer-assisted) proof for $N = 5$ was presented only recently[22]. Note also that twenty years ago Stephen Smale included it into the list of eighteen 'new Hilbert problems' for the 21st century

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Unfortunately – and this is another point Rafael often stressed – *physics is forgotten at that!* They quote, for instance, *Tamme's problem* in botany but not Thomson. The *plum-pudding model* was wrong, of course, but still physics was the original inspiration here!

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Universal optimality by Cohen and Kumar

Consider N points $\{x_i\}_{i=1}^N$ living on the unit sphere S^2. They form an *M-spherical design* if for any polynomial $x \mapsto p(x)$ on $\mathbb{R}^3$ of total degree M the equivalence one has $\int_{S^2} p(x)\mathrm{d}x = \frac{1}{N}\sum_i^N p(x_i)$ holds

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- $N = 2$, *antipodal points*
- $N = 3$, *simplex* with inner product $-1/2$,
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Remark: The remaining Platonic solids, *cube* and *dodekahedron*, do not qualify for universality having m=3 and 4, respectively. Note that they *do not represent Thomson problem solutions!*

Application to star leaky graphs

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Lemma

Consider an N-arm star with edges of length $L \in (0, \infty]$ determined by unit vectors $\{\bar{\gamma}_i\}_{i=1}^N$, and let $\{\bar{\sigma}_i\}_{i=1}^N$ corresponds to a sharp-configuration star. Then we have

$$\sum_{i,j\; i\neq j} T_{\kappa;s,t}(|\bar{\gamma}_i - \bar{\gamma}_j|^2) \geq \sum_{i,j\; i\neq j} T_{\kappa;s,t}(|\bar{\sigma}_i - \bar{\sigma}_j|^2)$$

for any $s, t \in [0, L]$ and the inequality is sharp unless the two stars are congruent. Here $T_{\kappa;s,t}(x) := \frac{\mathrm{e}^{-\kappa\sqrt{a+bx}}}{4\pi\sqrt{a+bx}}$ with $a = (s-t)^2$ and $b = st$

Application to star leaky graphs, continued

Next we use the fact that the largest eigenvalue of the Birman-Schwinger operator corresponding to a sharp-configuration star has the *maximum symmetry*, $\tilde{f}_\sigma = (f_\sigma, \dots, f_\sigma) \in \bigoplus_1^N L^2([0, L])$

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Then $\sup Q_{\kappa,\gamma} \geq (Q_{\kappa,\gamma}\tilde{f}_\sigma, \tilde{f}_\sigma) \geq \sup Q_{\kappa,\sigma}$ holds according to the above lemma, which allows us to make the following conclusion[24]

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Theorem

Assume that $N \in \{2,3,4,6,12\}$, then the ground state energy of the N-arm leaky star assumes the unique maximum for $\gamma = \sigma$, where σ is the corresponds to the appropriate sharp configuration listed above.

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Open questions

Ignoring various technical ones which appeared in the course of the presentation, there are deeper and more interesting questions, for instance

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¡Muchos años más felices, Rafael!