Isoperimetric problems for singular interactions in the plane

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Talk overview

**Motivation:** some classical and less classical isoperimetric results
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Point-interaction polygons: formulation of the problem

A geometric reformulation using Krein’s formula (or BS principle) and a convexity argument
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**Open questions**
Motivation

Isoperimetric problems are traditional in mathematical physics. Recall, e.g., the *Faber-Krahn inequality* for the Dirichlet Laplacian \(-\Delta^M_D\) in a compact \(M \subset \mathbb{R}^2\): among all regions with a fixed area the ground state is *uniquely minimized by the circle*,

\[
\inf \sigma(-\Delta^M_D) \geq \pi j_{0,1}^2 |M|^{-1}
\]

(we restrict to two dimensions in this talk, the analogous results naturally hold for any compact \(M \subset \mathbb{R}^d, d \geq 3\))
Motivation

Isoperimetric problems are traditional in mathematical physics. Recall, e.g., the Faber-Krahn inequality for the Dirichlet Laplacian $-\Delta^D_M$ in a compact $M \subset \mathbb{R}^2$: among all regions with a fixed area the ground state is uniquely minimized by the circle,

$$\inf \sigma(-\Delta^D_M) \geq \pi j_{0,1}^2 |M|^{-1}$$

(we restrict to two dimensions in this talk, the analogous results naturally hold for any compact $M \subset \mathbb{R}^d$, $d \geq 3$)

Another classical example is the PPW conjecture proved by Ashbaugh and Benguria: in the same situation we have

$$\frac{\lambda_2(M)}{\lambda_1(M)} \leq \left(\frac{j_{1,1}}{j_{0,1}}\right)^2$$
However, topology is important

If $M$ is not simply connected, rotational symmetry may again lead to an extremum but its nature can be different. Recall a *strip of fixed length and width* [E.-Harrell-Loss’99]

<ground state of\> < ground state of> whenever the strip is not a circular annulus
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![ground state of a non-annular strip](image)

whenever the strip is not a circular annulus

Another example is a *circular obstacle in circular cavity* [Harrell-Kröger-Kurata’01]

![ground state of a circular cavity with a circular obstacle](image)

whenever the obstacle is off center
Potential confinement

The topological distinction loses meaning if the particle is kept in a region by a (regular or singular) potential. To see what will happen we will analyze two models:
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First we take the simplest possible example where the confinement is due to a closed array of δ potentials, so the Hamiltonian can be written formally as

\[-\Delta + \tilde{\alpha} \sum_{j=1}^{N} \delta(x - y_j) \quad \text{in} \quad L^2(\mathbb{R}^2),\]

where the \(y_j\)'s are vertices of an equilateral polygon \(\mathcal{P}_N\).
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where the \(y_j\)'s are vertices of an equilateral polygon \(\mathcal{P}_N\).

Next we will consider an attractive δ potential supported by a closed loop \(\Gamma\) of fixed length, so formally we have

\[-\Delta - \alpha \delta(x - \Gamma) \quad \text{in} \quad L^2(\mathbb{R}^2)\]
Remarks

The two examples are related yet different in the character of the coupling, due the codimension of the interaction support. Roughly speaking, the 2D point interactions are a lot “more singular”
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- There are extensions to *higher dimension*, which will mentioned later at appropriate places
A preliminary: 2D point interactions

Fixing the site $y$ and “coupling constant” $\alpha$ we define them by b.c. which change \textit{locally} the domain of $-\Delta$: we require

$$\psi(x) = -\frac{1}{2\pi} \log |x - y| L_0(\psi, y) + L_1(\psi, y) + O(|x - y|),$$

where the generalized b.v. $L_0(\psi, y)$ and $L_1(\psi, y)$ satisfy

$$L_1(\psi, y) + 2\pi \alpha L_0(\psi, y) = 0, \quad \alpha \in \mathbb{R}.$$
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In this way we define our Hamiltonian \(-\Delta_{\alpha,\mathcal{P}_N}\) in \(L^2(\mathbb{R}^2)\) with \(N\) point interactions. We have \(\sigma_{\text{disc}}(-\Delta_{\alpha,\mathcal{P}_N}) \neq \emptyset\), i.e.

\[
\epsilon_1 \equiv \epsilon_1(\alpha, \mathcal{P}_N) := \inf \sigma(-\Delta_{\alpha,\mathcal{P}_N}) < 0,
\]

which is always true in two dimensions – cf. [AGHH’88, 05]
The point-interaction result

**Theorem [E.’05]:** Under the stated conditions, $\epsilon_1(\alpha, \mathcal{P}_N)$ is for fixed $\alpha$ and $\ell$ *locally sharply maximized* by a regular polygon, $\mathcal{P}_N = \tilde{\mathcal{P}}_N$. 
The point-interaction result

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Proof will be reduced to the following geometric problem:
Let $\mathcal{P}_N$ be an equilateral polygon. Given a fixed integer $m = 2, \ldots, \lceil \frac{1}{2} N \rceil$ we denote by $\mathcal{D}_m$ the sum of lengths of all $m$-diagonals, i.e. we put $\mathcal{D}_m := \sum_{i=1}^{N} |y_i - y_{i+m}|$

$D^1_{N,\ell}(m)$ The quantity $\mathcal{D}_m$ is, in the set of equilateral polygons $\mathcal{P}_N \subset \mathbb{R}^2$ with a fixed edge length $\ell > 0$, uniquely maximized by $\tilde{\mathcal{D}}_m$ referring to the (family of) regular polygon(s) $\tilde{\mathcal{P}}_N$. 
Geometric reformulation

By Krein's formula, the spectral condition is reduced to an algebraic problem. Using $k = i\kappa$ with $\kappa > 0$, we find the ev's of our operator from

$$\det \Gamma_k = 0 \quad \text{with} \quad (\Gamma_k)_{ij} := (\alpha - \xi^k)\delta_{ij} - (1 - \delta_{ij})g^k_{ij},$$

where the off-diagonal elements are $g^k_{ij} := G_k(y_i - y_j)$, or equivalently

$$g^k_{ij} = \frac{1}{2\pi} K_0(\kappa|y_i - y_j|)$$

and the regularized Green's function at the interaction site is

$$\xi^k = -\frac{1}{2\pi} \left( \ln \frac{\kappa}{2} + \gamma_E \right)$$
Geometric reformulation, continued

The ground state refers to the point where the lowest ev of $\Gamma_{i\kappa}$ vanishes. Using smoothness and monotonicity of the $\kappa$-dependence we have to check that

$$\min \sigma(\Gamma_{i\kappa_1}) < \min \sigma(\tilde{\Gamma}_{i\kappa_1})$$

holds locally for $\mathcal{P}_N \neq \tilde{\mathcal{P}}_N$, where $-\tilde{\kappa}_1^2 := \epsilon_1(\alpha, \tilde{\mathcal{P}}_N)$.
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There is a \textit{one-to-one relation} between an ef $c = (c_1, \ldots, c_N)$ of $\Gamma_{i\kappa}$ at that point and the corresponding ef of $-\Delta_{\alpha, \mathcal{P}_N}$ given by $c \leftrightarrow \sum_{j=1}^{N} c_j G_{ij}(\cdot - y_j)$, up to normalization. In particular, the lowest ev of $\tilde{\Gamma}_{i\tilde{\kappa}_1}$ corresponds to the eigenvector $\tilde{\phi}_1 = N^{-1/2}(1, \ldots, 1)$. Hence

$$\min \sigma(\tilde{\Gamma}_{i\tilde{\kappa}_1}) = (\tilde{\phi}_1, \tilde{\Gamma}_{i\tilde{\kappa}_1} \tilde{\phi}_1) = \alpha - \xi^{i\tilde{\kappa}_1} - \frac{2}{N} \sum_{i<j} \tilde{g}_{ij}^{i\tilde{\kappa}_1}$$
Geometric reformulation, continued

On the other hand, we have $\min \sigma(\Gamma \tilde{\kappa}_1) \leq (\tilde{\phi}_1, \Gamma \tilde{\kappa}_1 \tilde{\phi}_1)$, and therefore it is sufficient to check that

$$\sum_{i<j} G_{i\kappa}(y_i - y_j) > \sum_{i<j} G_{i\kappa}(\tilde{y}_i - \tilde{y}_j)$$

holds for all $\kappa > 0$ and $\mathcal{P}_N \neq \tilde{\mathcal{P}}_N$. 
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On the other hand, we have \( \min \sigma(\Gamma_{i\kappa_1}) \leq (\tilde{\phi}_1, \Gamma_{i\kappa_1}\tilde{\phi}_1) \), and therefore it is sufficient to check that

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holds for all \( \kappa > 0 \) and \( \mathcal{P}_N \neq \tilde{\mathcal{P}}_N \). Call \( \ell_{ij} := |y_i - y_j| \) and \( \tilde{\ell}_{ij} := |\tilde{y}_i - \tilde{y}_j| \) and define \( F : (\mathbb{R}_+)^{N(N-3)/2} \to \mathbb{R} \) by

\[
F(\{\ell_{ij}\}) := \sum_{m=2}^{[N/2]} \sum_{|i-j|=m} \left[ G_{i\kappa}(\ell_{ij}) - G_{i\kappa}(\tilde{\ell}_{ij}) \right];
\]

Using the convexity of \( G_{i\kappa}(\cdot) \) for a fixed \( \kappa > 0 \) we get

\[
F(\{\ell_{ij}\}) \geq \sum_{m=2}^{[N/2]} \nu_m \left[ G_{i\kappa} \left( \frac{1}{\nu_m} \sum_{|i-j|=m} \ell_{ij} \right) - G_{i\kappa}(\tilde{\ell}_{1,1+m}) \right],
\]

where \( \nu_n \) is the number of the appropriate diagonals.
Since $G_{ik}(\cdot)$ is also \textit{monotonously decreasing} in $(0, \infty)$, we need

$$\tilde{\ell}_{1,m+1} \geq \frac{1}{\nu_n} \sum_{|i-j|=m} \ell_{ij}$$

with the sharp inequality for at least one $m$ if $\mathcal{P}_N \neq \tilde{\mathcal{P}}_N$. In this way the problem becomes \textbf{purely geometric}
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The claim we made is then implied by the following result:

**Proposition:** The property \( D_{N,\ell}^1(m) \) holds \textit{locally} for any \( m = 2, \ldots, [\frac{1}{2}N] \)
Geometric reformulation, continued

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The claim we made is then implied by the following result:

**Proposition:** The property $D^{1}_{N,\ell}(m)$ holds locally for any $m = 2, \ldots, \lfloor \frac{1}{2}N \rfloor$

**Remark:** The argument carries through for point interactions in $\mathbb{R}^3$ because the Green’s function is again convex and monotonous.
We are looking for constrained local maxima of the function

\[
f_m : f_m(y_1, \ldots, y_N) = \frac{1}{N} \sum_{i=1}^{N} |y_i - y_{i+m}|
\]

with \( g_i(y_1, \ldots, y_n) := \ell - |y_i - y_{i+1}| = 0, \ i = 1, \ldots, N \). There are in fact \((N - 2)(d - 1) - 1\) independent variables because \(2d - 1\) parameters are related to Euclidean transformations.
Local validity of $D_{N, \ell}^1(m)$

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It is straightforward to check that $\nabla_j K_m(y_1, \ldots, y_N)$ vanish for a regular polygon, $K_m := f_m + \sum_{r=1}^{N} \lambda_r g_r$, with all the Lagrange multipliers taking the same value

$$\lambda = \frac{\sigma_m}{N \gamma_m} \quad \text{with} \quad \sigma_m := \frac{\sin^2 \frac{\pi m}{N}}{\sin^2 \frac{\pi}{N}}, \quad \gamma_m := \ell^{-1} |\tilde{y}_j - \tilde{y}_j \pm m|$$
Local validity of $D_{N,\ell}(m)$, continued

Negative definiteness of the Hessian needs more computation. A simple estimate then shows that it is sufficient to establish negative definiteness of the form

$$\xi \mapsto S_m[\xi] := \sum_j \left\{ |\xi_j - \xi_{j+m}|^2 - \sigma_m |\xi_j - \xi_{j+1}|^2 \right\}$$

on $\mathbb{R}^{2N}$ (the case $m = 2$ needs an additional argument)
Local validity of $D^1_{N,\ell}(m)$, continued

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on $\mathbb{R}^{2N}$ (the case $m = 2$ needs an additional argument).

The two parts can be simultaneously diagonalized; using their ev’s one rewrites the condition as the inequality

$$U_{m-1} \left( \cos \frac{\pi}{N} \right) > \left| U_{m-1} \left( \cos \frac{\pi r}{N} \right) \right|, \ r = 2, \ldots, m - 1,$$

for Chebyshev polynomials of the second kind which can be checked directly □
Attractive $\delta$ loops

To formulate the continuous analogue we have first to give meaning the formal operator

$$H_{\alpha, \Gamma} = -\Delta - \alpha \delta(x - \Gamma), \quad \alpha > 0,$$

in $L^2(\mathbb{R}^2)$, where $\Gamma$ is a loop in the plane; we suppose that it has no zero-angle self-intersections.
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$H_{\alpha,\Gamma}$ can be naturally associated with the quadratic form,

$$\psi \mapsto \|\nabla \psi\|_{L^2(\mathbb{R}^2)}^2 - \alpha \int_{\Gamma} |\psi(x)|^2 dx,$$

which is closed and below bounded in $W^{1,2}(\mathbb{R}^2)$; the second term makes sense in view of Sobolev embedding. This definition also works for various “wilder” sets $\Gamma$. 
Definition by boundary conditions

If \( \Gamma \) is \textit{piecewise smooth with no cusps} we can use an \textit{alternative definition} by boundary conditions: \( H_{\alpha, \Gamma} \) acts as \(-\Delta\) on functions from \( W^{2,1}_{\text{loc}}(\mathbb{R}^2 \setminus \Gamma) \), which are continuous and exhibit a normal-derivative jump,

\[
\left. \frac{\partial \psi}{\partial n}(x) \right|_+ - \left. \frac{\partial \psi}{\partial n}(x) \right|_- = -\alpha \psi(x)
\]
Definition by boundary conditions

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$$\frac{\partial \psi}{\partial n}(x) \bigg|_+ - \frac{\partial \psi}{\partial n}(x) \bigg|_- = -\alpha \psi(x)$$

Remarks:

- this definition has an illustrative meaning which corresponds to a $\delta$ potential in the cross cut of $\Gamma$

- using the quadratic form associated with $H_{\alpha, \Gamma}$ one can check directly that the discrete spectrum is not void for any $\alpha > 0$; one has, of course, $\sigma_{\text{ess}}(H_{\alpha, \Gamma}) = [0, \infty)$
The loop result

Let $\Gamma : [0, L] \rightarrow \mathbb{R}^2$ be a closed curve, $\Gamma(0) = \Gamma(L)$, parametrized by its arc length, which is $C^1$-smooth, piecewise $C^2$, and has no cusps. We will always consider classes of Euclidean transforms of $\Gamma$; it is clear that the circle class, $\mathcal{C} := \{ ((L/2\pi) \cos s, (L/2\pi) \sin s) : s \in [0, L] \}$, belongs to this family.
The loop result

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**Theorem [E.’05]:** Within the specified class of curves,

$$
\epsilon_1 \equiv \epsilon_1(\alpha, \Gamma) := \inf \sigma \left( H_{\alpha, \Gamma} \right)
$$

is for any fixed $\alpha > 0$ and $L > 0$ locally sharply maximized by a circle, $\Gamma = C$. 
We employ the generalized Birman-Schwinger principle [BEKŠ′94]. One starts from the free resolvent $R^k_0$ which is an integral operator in $L^2(\mathbb{R}^2)$ with the kernel

$$G_k(x-y) = \frac{i}{4} H^{(1)}_0(k|x-y|)$$
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$$G_k(x-y) = \frac{i}{4} H_0^{(1)}(k|x-y|)$$

Then we introduce embedding operators associated with $R^k_0$ for measures $\mu, \nu$ which are the Dirac measure $m$ supported by $\Gamma$ and the Lebesgue measure $dx$ on $\mathbb{R}^2$; by $R^k_{\nu,\mu}$ we denote the integral operator from $L^2(\mu)$ to $L^2(\nu)$ with the kernel $G_k$, i.e. we suppose that

$$R^k_{\nu,\mu} \phi = G_k * \phi \mu$$

holds $\nu$-a.e. for all $\phi \in D(R^k_{\nu,\mu}) \subset L^2(\mu)$.
Proposition [BEKŠ’94, Posilicano’04]:

(i) There is $\kappa_0 > 0$ s.t. $I - \alpha R^{i\kappa}_{m,m}$ on $L^2(m)$ has a bounded inverse for $\kappa \geq \kappa_0$.

(ii) Let $\text{Im } k > 0$ and $I - \alpha R^k_{m,m}$ be invertible with

$$R^k := R^k_0 + \alpha R^k_{dx,m} [I - \alpha R^k_{m,m}]^{-1} R^k_{m, dx}$$

from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ everywhere defined. Then $k^2$ belongs to $\rho(H_\alpha, \Gamma)$ and $(H_\alpha, \Gamma - k^2)^{-1} = R^k$.

(iii) $\dim \ker(H_\alpha, \Gamma - k^2) = \dim \ker(I - \alpha R^k_{m,m})$ for $\text{Im } k > 0$.

(iv) An ef of $H_\alpha, \Gamma$ associated with $k^2$ can be written as

$$\psi(x) = \int_0^L R^k_{dx,m}(x, s) \phi(s) \, ds,$$

where $\phi$ is the corresponding ef of $\alpha R^k_{m,m}$ with the ev one.
BS reformulation, continued

Putting $k = i\kappa$ with $\kappa > 0$ we look thus for solutions to the integral-operator equation

$$\mathcal{R}^{\kappa}_{\alpha, \Gamma} \phi = \phi, \quad \mathcal{R}^{\kappa}_{\alpha, \Gamma}(s, s') := \frac{\alpha}{2\pi} K_0(\kappa|\Gamma(s) - \Gamma(s')|),$$
onumber

on $L^2([0, L])$. The function $\kappa \mapsto \mathcal{R}^{\kappa}_{\alpha, \Gamma}$ is strictly decreasing in $(0, \infty)$ and $\|\mathcal{R}^{\kappa}_{\alpha, \Gamma}\| \to 0$ as $\kappa \to \infty$, hence we seek the point where the largest ev of $\mathcal{R}^{\kappa}_{\alpha, \Gamma}$ crosses one
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We observe that this ev is simple, since $\mathcal{R}_{\alpha,\Gamma}^\kappa$ is positivity improving and ergodic. The ground state of $H_{\alpha,\Gamma}$ is, of course, also simple. Using its rotational symmetry and the claim (iv) of the Proposition we find that the respective eigenfunction of $\tilde{\mathcal{R}}_{\alpha,\Gamma}^{\tilde{\kappa}_1}$ corresponding to the unit eigenvalue is constant; we can choose it as $\tilde{\phi}_1(s) = L^{-1/2}$. 

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Then we have

\[
\max \sigma(\mathcal{R}_{\alpha,\mathcal{C}}) = (\tilde{\phi}_1, & \mathcal{R}_{\alpha,\mathcal{C}} \tilde{\phi}_1) = \frac{1}{L} \int_0^L \int_0^L \mathcal{R}_{\alpha,\mathcal{C}}(s, s') \, ds \, ds',
\]

and on the other hand, for the same quantity referring to a general \( \Gamma \) a simple variational estimate gives

\[
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\]
Then we have

$$\max \sigma (\mathcal{R}_{\alpha, C}^{\tilde{\kappa}_1}) = (\tilde{\phi}_1, \mathcal{R}_{\alpha, C}^{\tilde{\kappa}_1} \tilde{\phi}_1) = \frac{1}{L} \int_0^L \int_0^L \mathcal{R}_{\alpha, C}^{\tilde{\kappa}_1}(s, s') \, ds \, ds'$$

and on the other hand, for the same quantity referring to a general $\Gamma$ a simple variational estimate gives

$$\max \sigma (\mathcal{R}_{\alpha, \Gamma}^{\tilde{\kappa}_1}) \geq (\tilde{\phi}_1, \mathcal{R}_{\alpha, \Gamma}^{\tilde{\kappa}_1} \tilde{\phi}_1) = \frac{1}{L} \int_0^L \int_0^L \mathcal{R}_{\alpha, \Gamma}^{\tilde{\kappa}_1}(s, s') \, ds \, ds'$$

Hence it is sufficient to show that

$$\int_0^L \int_0^L K_0(\kappa|\Gamma(s)-\Gamma(s')|) \, ds \, ds' \geq \int_0^L \int_0^L K_0(\kappa|C(s)-C(s')|) \, ds \, ds'$$

holds for all $\kappa > 0$ and $\Gamma$ in the vicinity of $C$. 


UAB05 Conference “Differential Equations and Mathematical Physisc”; Birmingham, Al., April 1, 2005 – p.21/40
Convexity argument

By a simple change of variables the claim is equivalent to positivity of the functional

\[ F_\kappa(\Gamma) := \int_0^{L/2} du \int_0^L ds \left[ K_0(\kappa |\Gamma(s+u) - \Gamma(s)|) - K_0(\kappa |C(s+u) - C(s)|) \right] \]

the \( s \)-independent second term is equal to \( K_0(\frac{\kappa L}{\pi} \sin \frac{\pi u}{L}) \)
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the \( s \)-independent second term is equal to \( K_0\left(\frac{\kappa L}{\pi} \sin \frac{\pi u}{L}\right) \)

The (strict) convexity of \( K_0 \) yields by means of Jensen inequality the estimate

\[ \frac{1}{L} F_{\kappa}(\Gamma) \geq \int_0^{L/2} \left[ K_0 \left( \frac{\kappa}{L} \int_0^L \, |\Gamma(s+u) - \Gamma(s)| \, ds \right) - K_0 \left( \frac{\kappa L}{\pi} \sin \frac{\pi u}{L} \right) \right] \, du , \]

where the inequality is sharp unless \( \int_0^L |\Gamma(s+u) - \Gamma(s)| \, ds \) is independent of \( s \).
Monotonicity argument

Finally, we observe that $K_0$ is decreasing in $(0, \infty)$, hence it is sufficient to check the inequality

$$
\int_0^L |\Gamma(s+u) - \Gamma(s)| \, ds \leq \frac{L^2}{\pi} \sin \frac{\pi u}{L}
$$

for all $u \in (0, \frac{1}{2}L]$ and furthermore, to show that is sharp unless $\Gamma$ is a circle.
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**Remark:** There was nothing *local* so far, hence proving the above inequality for all $\Gamma$ would give the global result. Likewise, we have not used the $C^2$ smoothness
Both geometric reformulations have a common feature: for polygons we sum diagonal lengths between vertices whose indices differ by a fixed $m$, for a loop we integrate chord lengths between points separated by a fixed arc length $u$. 
Mean-chord inequalities

Consider a wider family of inequalities – without knowing whether they are valid. Let \( \Gamma : [0, L] \to \mathbb{R}^2 \) be again a loop in the plane, with unspecified regularity properties. Take all the arcs of \( \Gamma \) having length \( u \in (0, \frac{1}{2} L] \) and write

\[
C^p_L(u) : \quad \int_0^L |\Gamma(s+u) - \Gamma(s)|^p \, ds \leq \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L} , \quad p > 0 ,
\]

\[
C^{-p}_L(u) : \quad \int_0^L |\Gamma(s+u) - \Gamma(s)|^{-p} \, ds \geq \frac{\pi^p L^{1-p}}{\sin^p \frac{\pi u}{L}} , \quad p > 0 .
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\]

A discrete counterpart for an equilateral polygon \( P_N \) of \( N \) vertices \( \{y_n\} \), side length \( \ell > 0 \), and \( m = 1, \ldots, \lfloor \frac{1}{2}N \rfloor \) reads

\[
D^p_{N,\ell}(m) : \quad \sum_{n=1}^{N} |y_{n+m} - y_n|^p \leq \frac{N \ell^p \sin^p \frac{\pi m}{N}}{\sin^p \frac{\pi}{N}}, \quad p > 0,
\]

\[
D^{-p}_{N,\ell}(m) : \quad \sum_{n=1}^{N} |y_{n+m} - y_n|^{-p} \geq \frac{N \sin^p \frac{\pi}{N}}{\ell^p \sin^p \frac{\pi m}{N}}, \quad p > 0.
\]
Observations

The right-hand sides correspond to the cases with maximum symmetry, i.e. to the circle and regular polygon $\tilde{P}_N$, respectively.
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- If $p = 0$ the inequalities turn into trivial identities.
- By scaling one can put, for instance, $L = 1$ and $\ell = 1$ without loss of generality.
- In the polygon case it is clear that the claim may not be true for $p > 2$ as the example of a rhomboid shows: $D^p_{4,\ell}(2)$ is equivalent to $\sin^p \phi + \cos^p \phi \leq 2^{1-(p/2)}$ for $0 < \phi < \pi$. 


Properties and conjecture

Using convexity of \( x \mapsto x^\alpha \) in \((0, \infty)\) for \( \alpha > 1 \) we get

**Proposition:** \( C_p^L(u) \Rightarrow C_{p'}^L(u) \) and \( D_{N,\ell}^p(m) \Rightarrow D_{N,\ell}^{p'}(m) \) if \( p > p' > 0 \)
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Furthermore, Schwarz inequality implies

**Proposition:** $C^p_L(u) \Rightarrow C^{-p}_L(u)$ and $D^p_{N, \ell}(m) \Rightarrow D^{-p}_{N, \ell}(m)$

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**Proposition:** $C^p_L(u) \Rightarrow C^{-p}_L(u)$ and $D^p_{N,\ell}(m) \Rightarrow D^{-p}_{N,\ell}(m)$ for any $p > 0$

**Conjecture:** We expect the above inequalities to be valid for any $p \leq 2$, without substantial regularity restrictions in the continuous case.
What is known for $D_{N,\ell}^{p}(m)$?

We have shown that $D_{N,\ell}^{1}(m)$ holds \textit{locally} for any $m = 2, \ldots, [\frac{1}{2}N]$, i.e. in the vicinity of the regular polygon, and consequently, $D_{N,\ell}^{\pm p}(m)$ holds locally for any $p \in (0, 1]$.
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As for the \emph{global validity} we have a particular result:

**Proposition:** $D_{N,\ell}^1(2)$ holds \emph{globally}, and so does $D_{N,\ell}^{\pm p}(2)$ for each $p \in (0, 1]$
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We have shown that $D^1_{N,\ell}(m)$ holds locally for any $m = 2, \ldots, \lceil \frac{1}{2} N \rceil$, i.e. in the vicinity of the regular polygon, and consequently, $D^{\pm p}_{N,\ell}(m)$ holds locally for any $p \in (0, 1]$

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**Proof:** Call $\beta_i$ the “bending angle” at $i$-th vertex, then the mean length of the 2-diagonals is $M_2 = \frac{2\ell}{N} \sum_{i=1}^N \cos \frac{\beta_i}{2}$. Using strict convexity of the function $u \mapsto -\cos \frac{u}{2}$ in $(-\pi, \pi)$ together with $\sum_{i=1}^N \beta_i = 2\pi w$, $w \in \mathbb{Z}$, we find

$$- \sum_{i=1}^N \cos \frac{\beta_i}{2} \geq -N \cos \left( \sum_{i=1}^N \frac{\beta_i}{2} \right) = -N \cos \frac{\pi}{N};$$

the inequality is sharp unless all the $\beta_i$'s are the same $\square$
$C^p_L(u)$ in terms of curvature

Under our regularity assumption we can characterize $\Gamma$ by its (signed) curvature $\gamma := \dot{\Gamma}_2 \ddot{\Gamma}_1 - \dot{\Gamma}_1 \ddot{\Gamma}_2$ which is piecewise continuous in $[0, L]$. Up to Euclidean transf’s we have

$$\Gamma(s) = \left( \int_0^s \cos \beta(s') \, ds', \int_0^s \sin \beta(s') \, ds' \right),$$

where $\beta(s) := \int_0^s \gamma(s') \, ds'$ is bending angle relative to $s = 0$.
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To ensure that the curve is closed, we have to require

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To ensure that the curve is closed, we have to require

$$\int_0^L \cos \beta(s') \, ds' = \int_0^L \sin \beta(s') \, ds' = 0$$

The left-hand side of $C^p_L(u)$ can be now rewritten as

$$c^p_{\Gamma}(u) = \int_0^L ds \left[ \int_s^{s+u} ds' \int_s^{s+u} ds'' \cos(\beta(s') - \beta(s'')) \right]^{p/2}$$
Proof of $C^2_L(u)$

It is sufficient to check that $c^2_T(u)$ is maximized by the circle, i.e. by $\beta(s) = \frac{2\pi s}{L}$. Rearranging the integrals we get

$$c^2_T(u) = \int_0^L ds' \int_{s' - u}^{s' + u} ds'' [u - |s' - s''|] \cos(\beta(s') - \beta(s''))$$
Proof of $C_L^2(u)$

It is sufficient to check that $c_T^2(u)$ is maximized by the circle, i.e. by $\beta(s) = \frac{2\pi s}{L}$. Rearranging the integrals we get

$$c_T^2(u) = \int_0^L ds' \int_{s'-u}^{s'+u} ds'' \left[ u - |s' - s''| \right] \cos(\beta(s') - \beta(s''))$$

Next we change the integration variables to $x := s' - s''$ and $z := \frac{1}{2}(s' + s'')$, and use the even parity of the functions involved to obtain

$$c_T^2(u) = 2 \int_0^u dx \left( u - x \right) \int_0^L dz \cos \left( \int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} \gamma(s) \, ds \right)$$
A partial global result

In an analogy with $D_{N,\ell}^{1}(2)$ we can get a global result for $u$ small enough:

**Proposition:** Let $\Gamma$ have no self-intersections and the inequality $\beta(z + \frac{1}{2}u) - \beta(z - \frac{1}{2}u) \leq \frac{1}{2}\pi$ is valid for all $z \in [0, L]$, then $C_{L}^{2}(u)$ holds.
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**Proof:** We employ concavity of cosine in $(0, \frac{1}{2}\pi)$ obtaining

$$c_\Gamma^2(u) \leq 2L \int_0^u dx (u - x) \cos \left( \frac{1}{L} \int_0^L dz \int_{z - \frac{1}{2}x}^{z + \frac{1}{2}x} \gamma(s) ds \right)$$

$$= 2L \int_0^u dx (u - x) \cos \frac{2\pi x}{L} = \frac{L^3}{\pi^2} \sin^2 \frac{\pi u}{L},$$

since $\int_0^L \gamma(s) ds = \pm 2\pi$. The function $z \mapsto \int_{z - \frac{1}{2}x}^{z + \frac{1}{2}x} \gamma(s) ds$ is constant for $x \in (0, u)$ iff $\gamma(\cdot)$ is constant, hence the circle gives a sharp maximum. $\square$
Local validity of $C^2_L(u)$

**Proposition:** If $\Gamma$ is $C^1$, piecewise $C^2$, the inequality $C^2_L(u)$ holds locally for any $L > 0$ and $u \in (0, \frac{1}{2}L]$, and consequently, $C^{\pm p}_L(u)$ holds locally for any $p \in (0, 2]$. 

Proof: Gentle deformations of $\Gamma$ can be characterized by $(s) = 2L + g(s)$; where $g$ is a piecewise continuous function, small in the sense that $k g k_1 \leq L^1$ and satisfying the condition $\int L g(s) \, ds = 0$. We employ the expansion

$$
\cos 2x L z + 1 2 x z 1 2 x g(s) \, ds
$$

where the error term is a shorthand for $O(k L g^3 k)$.
Local validity of $C_L^2(u)$

**Proposition:** If $\Gamma$ is $C^1$, piecewise $C^2$, the inequality $C_L^2(u)$ holds locally for any $L > 0$ and $u \in (0, \frac{1}{2}L]$, and consequently, $C_L^{p}(u)$ holds locally for any $p \in (0, 2]$

**Proof:** Gentle deformations of $\mathcal{C}$ can be characterized by

$$\gamma(s) = \frac{2\pi}{L} + g(s),$$

where $g$ is a piecewise continuous function, small in the sense that $\|g\|_\infty \ll L^{-1}$ and satisfying the condition

$$\int_0^L g(s) \, ds = 0.$$ 

We employ the expansion

$$\cos \frac{2\pi x}{L} - \sin \frac{2\pi x}{L} \int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s) \, ds - \frac{1}{2} \cos \frac{2\pi x}{L} \left( \int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s) \, ds \right)^2 + \mathcal{O}(g^3),$$

where the error term is a shorthand for $\mathcal{O}(\|Lg\|_\infty^3)$.
Proof, continued

Substituting into the expression for \( c^2_1(u) \) we find that the term linear in \( g \) vanishes, because

\[
\int_0^L dz \int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s) \, ds = \int_0^L ds \, g(s) \int_{s-\frac{1}{2}x}^{s+\frac{1}{2}x} dz = 0 ,
\]
Substituting into the expression for $c^2_\Gamma(u)$ we find that the term linear in $g$ vanishes, because

$$\int_0^L dz \int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s) \, ds = \int_0^L ds g(s) \int_{s-\frac{1}{2}x}^{s+\frac{1}{2}x} dz = 0,$$

Hence the deformation shows in the 2nd order term only,

$$c^2_\Gamma(u) = \frac{L^3}{\pi^2} \sin^2 \frac{\pi u}{L} - I_g(u) + O(g^3),$$

where

$$I_g(u) := \int_0^u dx (u-x) \cos \frac{2\pi x}{L} \int_0^L dz \left( \int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s) \, ds \right)^2$$

and we need to show that $I_g(u) > 0$ unless $g = 0$ identically. Notice that for $u \leq \frac{1}{4}L$ this property holds trivially.
Proof, continued

For \( u \in \left( \frac{1}{4}L, \frac{1}{2}L \right] \) we notice that \( g \) is periodic and piecewise \( C^0 \), so we write it as Fourier series with zero term missing,

\[
g(s) = \sum_{n=1}^{\infty} \left( a_n \sin \frac{2\pi ns}{L} + b_n \cos \frac{2\pi ns}{L} \right),
\]

where \( \sum_n (a_n^2 + b_n^2) < \infty \) (and small).
Proof, continued

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where \( \sum_n (a_n^2 + b_n^2) < \infty \) (and small). We have

\[
\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s) \, ds = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( a_n \sin \frac{2\pi nz}{L} + b_n \cos \frac{2\pi nz}{L} \right) \sin \frac{\pi nx}{L},
\]

so using orthogonality of the Fourier basis one gets

\[
I_g(u) = \int_0^u dx \, (u - x) \cos \frac{2\pi x}{L} \sum_{n=1}^{\infty} \frac{L^3}{2\pi^2} \frac{a_n^2 + b_n^2}{n^2} \sin \frac{\pi nx}{L}.
\]
Summation and integration can be interchanged giving

\[ I_g(u) = \frac{L^5}{2\pi^4} \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{n^2} F_n \left( \frac{\pi u}{L} \right), \]

where

\[ F_n(v) := \int_0^v (v - y) \cos 2y \sin ny \, dy. \]
Proof, continued

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\[ F_n(v) := \int_0^v (v - y) \cos 2y \sin ny \, dy. \]

These integrals are equal to

\[ F_1(v) = \frac{1}{18} (9 \sin v - \sin 3v - 6v), \]
\[ F_2(v) = \frac{1}{32} (4v - \sin 4v), \]
\[ F_n(v) = \frac{nv}{n^2 - 4} - \frac{\sin(n - 2)v}{2(n - 2)^2} - \frac{\sin(n + 2)v}{2(n + 2)^2}, \quad n \geq 3. \]

It is easy to see that \( F_n(v) > 0 \) for \( v > 0 \) and \( n \geq 2 \) and \( F_1(v) > 0 \) in the interval \((0, \frac{\pi}{2})\). Thus we have found that \( I_g(u) > 0 \) unless all the coefficients \( a_n, b_n \) are zero. \( \square \)
Remark

One may wonder what happened with the \textit{closedness requirement}, \( \int_0^L \cos \beta(s') \, ds' = \int_0^L \sin \beta(s') \, ds' = 0 \). We proved the claim using the weaker property \( \beta(0) = \beta(L) \). This is possible \textit{for small deformations only}!
Remark

One may wonder what happened with the *closedness requirement*, \( \int_0^L \cos \beta(s') \, ds' = \int_0^L \sin \beta(s') \, ds' = 0 \). We proved the claim using the weaker property \( \beta(0) = \beta(L) \). This is possible *for small deformations only!*

As an illustration, consider \( \Gamma \) in the form of an “overgrown paperclip” which satisfies the condition \( \beta(0) = \beta(L) \) but not the *closedness requirement*. Making the U-turns small one can get \( c_\Gamma^2 \left( \frac{1}{2} L \right) \) *arbitrarily close to* \( \frac{1}{3} L^3 \) which is, of course, larger than \( L^3 / \pi^2 \).
Global validity of $\mathcal{C}^2_L(u)$: an example

Let $\Gamma$ be a curve consisting of two circular segments of radius $R > \frac{L}{4\pi}$, i.e. it is given by the equations

$$\left(x \pm R \cos \frac{L}{2R}\right)^2 + y^2 = R^2 \quad \text{for} \quad \pm x \geq 0$$

being “lens-shaped” for $R > \frac{L}{2\pi}$, “apple-shaped” for $\frac{L}{4\pi} < R < \frac{L}{2\pi}$ “apple-shaped” and a circle for $R = \frac{L}{2\pi}$
Example, continued

It is straightforward exercise to compute

$$c_\Gamma^2(u) = 8R^3 \left\{ \frac{L}{2R} \sin^2 \frac{u}{2R} + 4 \left( \frac{u}{2R} \cos \frac{u}{2R} - \sin \frac{u}{2R} \right) \cos \frac{L}{4R} \cos \frac{L - 2u}{4R} \right\}$$

Let us plot $c_\Gamma^2(u) \left( \frac{L^3}{\pi^2} \sin^2 \frac{\pi u}{L} \right)^{-1}$ for $L = 1$ w.r.t. $R$ and $u$
Example, continued

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Some open questions

- Prove $D^2_{N,\ell}(m)$, locally and globally
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- Prove $D_{N,\ell}^2(m)$, locally and globally
- Prove *global* validity of $C^p_L(u)$
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- Prove \textit{higher-dimensional analogues} of these inequalities for loops in $\mathbb{R}^d$ (notice that the local proof of $D^1_{N,\ell}(m)$ works for polygons in any $\mathbb{R}^d$)
Some open questions

- Prove $D^2_{N,\ell}(m)$, locally and globally
- Prove *global* validity of $C^p_L(u)$
- Prove *higher-dimensional analogues* of these inequalities for loops in $\mathbb{R}^d$ (notice that the local proof of $D^1_{N,\ell}(m)$ works for polygons in any $\mathbb{R}^d$)
- Prove *higher-dimensional analogues* of these inequalities for *codimension-one surfaces* in $\mathbb{R}^d$
Some open questions

- Prove $D_{N,\ell}^2(m)$, locally and globally

- Prove *global* validity of $C_L^p(u)$

- Prove *higher-dimensional analogues* of these inequalities for loops in $\mathbb{R}^d$ (notice that the local proof of $D_{N,\ell}^1(m)$ works for polygons in any $\mathbb{R}^d$)

- Prove *higher-dimensional analogues* of these inequalities for codimension-one surfaces in $\mathbb{R}^d$

- Find maximizers in classes not containing $C$ or $P_N$
Some open questions

- Prove $D^2_{N,\ell}(m)$, locally and globally
- Prove global validity of $C^p_L(u)$
- Prove higher-dimensional analogues of these inequalities for loops in $\mathbb{R}^d$ (notice that the local proof of $D^1_{N,\ell}(m)$ works for polygons in any $\mathbb{R}^d$)
- Prove higher-dimensional analogues of these inequalities for codimension-one surfaces in $\mathbb{R}^d$
- Find maximizers in classes not containing $C$ or $\mathcal{P}_N$
- Find maximizers if the interaction strength changes along the curve (or surface), so the problem ceases to be purely geometric
The talk was based on


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for more information see  http://www.ujf.cas.cz/~exner