Spectra of Laplacians in twisted tubes

Pavel Exner

in collaboration with *Hynek Kovařík*

exner@ujf.cas.cz

Doppler Institute

for Mathematical Physics and Applied Mathematics

Prague

Talk overview

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- Main result: slowing down the twist gives rise to a non-empty discrete spectrum

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- Extended twisting moves the essential spectrum: an example of a screw-shaped tube in $\mathbb{R}^3$
- Main result: slowing down the twist gives rise to a non-empty discrete spectrum
- Summary and outlook

Geometry & spectrum in Dirichlet tubes

Recall first some well-known facts:

Given an open, bounded and connected $\omega \subset \mathbb{R}^{d-1}$ consider *Dirichlet Laplacian* $-\Delta_D^{\omega\times\mathbb{R}}$ in the *straight tube* $\omega \times \mathbb{R}$.
Trivially, the spectrum is a.c. and equal to $[E_1, \infty)$ with the threshold $E_1 := \inf \sigma(-\Delta_D^{\omega})$

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On the other hand, *local geometric perturbations* such as

- a sharp break or several breaks
- a smooth bend with asymptotically vanishing curvature
- a local tube protrusion

give rise to a *non-empty discrete spectrum*, i.e. isolated eigenvalues below E_1

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- a lot more is known: weak-coupling asymptotics
- thin-tube asymptotics
- Lieb-Thirring-type inequalities
- many-body effects
- in addition, there are results about scattering, resonances, periodically curved tubes, etc.

Twisted straight tubes

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Given ω described above and a differentiable $\theta : \mathbb{R} \to \mathbb{R}$, we use $s \in \mathbb{R}$, $t := (t_2, t_3) \in \omega$ to define map $\mathcal{L} : \mathbb{R} \times \omega \to \mathbb{R}^3$ by

$$\mathcal{L}(s,t) = (s,\ t_2 \cos\theta(s) + t_3 \sin\theta(s),\ t_3 \cos\theta(s) - t_2 \sin\theta(s))$$

The image $\Omega := \mathcal{L}(\mathbb{R} \times \omega)$ is a tube in $\mathbb{R}^3$ which is twisted unless the function θ is constant.

We are interested in Dirichlet Laplacian on $L^2(\Omega)$, i.e. the s-a operator associated with the closed quadratic form

$$Q[\psi] := \int_\Omega |\nabla\psi|^2 \,\mathrm{d}s\,\mathrm{d}t\,, \quad \forall\, \psi \in D(Q) = \mathcal{H}_0^1(\Omega)$$

An alternative expression

To any radial vector $t \equiv (t_2, t_3) \in \mathbb{R}^2$ we associate the normal one, $\tau(t) := (t_3, -t_2)$ and use it to introduce the angular-derivative operator

$$\partial_\tau := t_3 \, \partial_2 - t_2 \, \partial_3$$

Given a bounded $\sigma : \mathbb{R} \to \mathbb{R}$, we consider the self-adjoint operator L_σ associated with the quadratic form

$$l_\sigma[\psi] := \|\partial_1 \psi - \sigma \, \partial_\tau \psi\|^2 + \|\partial_2 \psi\|^2 + \|\partial_3 \psi\|^2$$

with $\psi \in D(l_\sigma) := \mathcal{H}_0^1(\mathbb{R} \times \omega)$

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Choosing now, in particular, $\sigma = \dot\theta$ one can check by a straightforward calculation, using natural coordinate transformation, that this operator is unitarily equivalent to the Dirichlet Laplacian introduced above

A Hardy-type inequality

If $\sigma = 0$, of course, L_0 is Dirichlet Laplacian in the straight tube. Also the case of a tube with circular ω centered at the origin is trivial. In all the other situations we have the following result:

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Theorem [Ekholm-Kovařík-Krejčiřík'05]: Let ω be a bounded open connected subset of $\mathbb{R}^2$ which is *not* rotationally invariant. Let σ be a nonzero compactly supported continuous function with bounded derivatives. Then for all $\psi \in \mathcal{H}_0^1(\mathbb{R} \times \omega)$ and s_0 such that $\sigma(s_0) \neq 0$ we have

$$l_\sigma[\psi] - E_1 \|\psi\|^2 \;\geq\; c \int_{\mathbb{R}\times\omega} \frac{|\psi(s,t)|^2}{1 + (s - s_0)^2} \, \mathrm{d}s \, \mathrm{d}t$$

with $c > 0$ is independent of ψ but depending on s_0, σ and ω.

A Hardy-type inequality, continued

The inequality is proved in two steps: first one derives a "local" inequality for the operator L_σ over a finite piece of the tube where σ is nonzero; then the local result is "smeared out" by means of a classical one-dimensional Hardy inequality.

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Corollary [EHK'05]: Let Ω be a tube of non-circular cross section which is locally twisted. Then the spectrum $[E_1, \infty)$ of $-\Delta_D^\Omega$ is stable under sufficiently small bends.

Remark: In a similar way one can check spectral stability under other weak enough (attractive) perturbations (potentials, protrusions)

Nonlocally twisted tubes

We assumed above that $\dot{\theta}$ has a compact support so that $\sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = E_1 = \inf \sigma_{\mathrm{ess}}(-\Delta_D^{\mathbb{R}\times\omega})$

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This may not be true if the twist is infinitely extended; an example is *a screw-shaped tube* corresponding to a linear θ : we fix a positive constant β_0 and define Ω_0 by

$$\Omega_0 := \mathcal{L}_0(\mathbb{R}\times\omega)\,,$$

where

$$\mathcal{L}_0(s,t) := (s,\ t_2\cos(\beta_0 s)+t_3\sin(\beta_0 s),\ t_3\cos(\beta_0 s)-t_2\sin(\beta_0 s))\,.$$

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We will take Ω_0 as an unperturbed system and *conjecture* that a local *slowdown* of the twisting acts as *effective attractive interaction* which can give rise to bound states

The spectrum of H_0

We use again the unitary equivalence above, this time with uniformly rotating coordinate frame, in which $H_0 := -\Delta_D^{\Omega_0}$ acts on its domain in $L^2(\Omega_0)$ as

$$H_0 = -\partial_{t_2}^2 - \partial_{t_3}^2 + \left(-i\partial_s - i\,\beta_0\,(t_2\partial_{t_3} - t_3\partial_{t_2})\right)^2$$

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Since β_0 is independent of s we are able to employ a partial Fourier transformation $\mathcal{F}_s$ given by

$$(\mathcal{F}_s\,\psi)(p,t) = \hat{\psi}(p,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\,ps}\,\psi(s,t)\mathrm{d}s\,,$$

so for a suitably regular ψ we can rewrite the form as

$$Q_0[\hat{\psi}] = \int_{\mathbb{R}\times\omega} |\nabla_t\hat{\psi}|^2 + |i\,p\,\hat{\psi} + \beta_0\hat{\psi}'_\tau|^2\,\mathrm{d}p\,\mathrm{d}t$$

The spectrum of H_0, continued

Since $\mathcal{F}_s$ extends to a unitary operator on $L^2(\mathbb{R} \times \omega)$, the operator H_0 is equivalent to $\int_{\mathbb{R}}^{\oplus} h(p)\,\mathrm{d}p$ with the fibre

$$h(p) = -\partial_{t_2}^2 - \partial_{t_3}^2 + \left(p - i\,\beta_0(t_2\partial_{t_3} - t_3\partial_{t_2})\right)^2$$

on $L^2(\omega)$ subject to Dirichlet boundary conditions at $\partial\omega$.

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on $L^2(\omega)$ subject to Dirichlet boundary conditions at $\partial\omega$. Using polar coordinates (r, α) on ω we rewrite $h(p)$ as

$$h(p) = -\Delta_D^\omega + (p - i\,\beta_0\partial_\alpha)^2.$$

Since $h(p)$ is a sum of $-\Delta_D^\omega$ and a positive perturbation, by minimax principle its spectrum is purely discrete. Let us denote the eigenvalues of $h(p)$ by $\epsilon_n(p)$ and the respective eigenfunctions by $\psi_n(p)$, i.e.

$$h(p)\,\psi_n(p) = \epsilon_n(p)\,\psi_n(p)$$

The spectrum of H_0, continued

Lemma: $\epsilon_n(\cdot)$, $n \in \mathbb{N}$, is a real-analytic function of p and

$$\lim_{p \to \pm\infty} \epsilon_n(p) \to \infty$$

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Sketch of proof: The form associated with $h(0)$ defined on $\mathcal{H}^1_0(\Omega_0)$ is non-negative and closed, so $h(0)$ is self-adjoint on its natural domain denoted as $D(0)$. We formally expand the

$$h(p) = h(0) + p^2 - 2i\, p\, \beta_0\, \partial_\alpha$$

It is easy to check that $i\,\partial_\alpha$ is $h(0)$-bounded with the relative bound zero, so the domain of $h(p)$ coincides with $D(0)$ and $h(\cdot)\phi$ is analytic for every $\phi \in D(0)$. Then by [Kato'66] we have a type A operator family, and consequently, all the $\epsilon_n(\cdot)$ are real-analytic functions of p

The spectrum of H_0, continued

Put next $a := \sup_{t\in\omega} |t|$. For any $\varphi \in C_0^\infty(\omega)$ we have

$$|2p\,\beta_0\,\bar\varphi\,\partial_\alpha\varphi| \le p^2\,\frac{\beta_0^2}{\beta_0^2 + a^{-2}}\,|\varphi|^2 + (\beta_0^2 + a^{-2})\,|\partial_\alpha\varphi|^2\,,$$

which implies for $|p| \to \infty$

$$(\varphi,\, h(p)\,\varphi) \ge \frac{1}{1 + a^2\,\beta_0^2}\,p^2 \int_\omega |\varphi|^2\, r\,\mathrm{d}r\,\mathrm{d}\alpha \to 0\,. \qquad \square$$

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Clearly, spectral threshold of $h(0)$ cannot be lower than that of $-\Delta_D^\omega$. By [EKK'05] or lemma below the bound is sharp,

$$E := \inf\sigma(h(0)) > \inf\sigma\left(-\Delta_D^\omega\right)\,,$$

whenever ω is *not* rotationally symmetric

The spectrum of H_0, continued

We will show that $E = \inf \sigma(H_0)$. Denote by f the real-valued eigenfunction of $h(0)$ associated with $E = \epsilon_1(0)$,

$$h(0)f = -\Delta_D^\omega f - \beta_0^2\, \partial_\alpha^2 f = Ef$$

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Lemma: (a) f is strictly positive in ω

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Proof: To prove $f > 0$ we have to show that $\{e^{-t\, h(0)} :\ t \ge 0\}$ is positivity improving, i.e. $e^{-t\, h(0)} g > 0$ for any $g \ge 0$ and $t \ge 0$. Since $-\Delta_D^\omega$ commutes with ∂_α^2, we get

$$e^{-t\, h(0)} = e^{t\, \Delta_D^\omega}\, e^{t\, \beta_0^2\, \partial_\alpha^2}$$

Now $e^{t\, \beta_0^2\, \partial_\alpha^2}$ is positivity preserving for all $t > 0$ and $e^{t\, \Delta_D^\omega}$ is positivity improving; together this proves the first claim

The spectrum of H_0, continued

(b) Let B be the biggest circle centred at the origin s.t. $B \subset \overline{\omega}$ and $B^c \neq \emptyset$ its complement in $\overline{\omega}$. Since f satisfies Dirichlet b.c. on $\partial\omega$ and $f > 0$ inside, $|\partial_\alpha f| > 0$ is in a.e. point of $B^c \cap \partial\omega$, where $\partial\omega$ is not a part of a circle centred at the origin; by smoothness we find a positive-measure neighbourhood of $B^c \cap \partial\omega$ on which $|\partial_\alpha f| > 0$. $\square$

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Theorem [E.-Kovařík'05]: The spectrum of H_0 is purely absolutely continuous and covers the half-line $[E, \infty)$, where E is the lowest eigenvalue of $h(0)$

Proof: In view of the first lemma $\sigma(H_0)$ is a.c. and contains the interval $[E, \infty)$; it remains to show that

$$(-\infty,\, E) \cap \sigma(H_0) = \emptyset$$

The spectrum of H_0, continued

Since $f > 0$ in ω, we can decompose any $\psi \in C_0^\infty(\omega)$ as $\psi(s,t) = f(t)\varphi(s,t)$. Integrating by parts we get

$$Q_0[\psi] - E\,\|\psi\|^2 = \int_{\mathbb{R}\times\omega} \Big(f^2\,|\nabla_t \varphi|^2 - (\Delta_D^\omega f)\,f|\varphi|^2 + f^2\,|\partial_s\varphi|^2$$

$$+\beta_0\, f\partial_\alpha f\,(\partial_s\bar\varphi\,\varphi + \bar\varphi\,\partial_s\varphi) + \beta_0\, f^2(\partial_s\bar\varphi\,\partial_\alpha\varphi + \partial_\alpha\bar\varphi\,\partial_s\varphi)$$

$$+\beta_0^2\, f^2\,|\partial_\alpha\varphi|^2 - \beta_0^2\,(\partial_\alpha^2 f)\,f|\varphi|^2 - E\,f^2\,|\varphi|^2\Big)\,\mathrm{d}s\,\mathrm{d}t$$

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$$Q_0[\psi] - E\,\|\psi\|^2 = \int_{\mathbb{R}\times\omega} \Big(f^2\,|\nabla_t\varphi|^2 - (\Delta_D^\omega f)f|\varphi|^2 + f^2\,|\partial_s\varphi|^2 + \beta_0\, f\partial_\alpha f(\partial_s\bar\varphi\,\varphi + \bar\varphi\,\partial_s\varphi) + \beta_0\, f^2(\partial_s\bar\varphi\,\partial_\alpha\varphi + \partial_\alpha\bar\varphi\,\partial_s\varphi) + \beta_0^2\, f^2\,|\partial_\alpha\varphi|^2 - \beta_0^2\,(\partial_\alpha^2 f)f|\varphi|^2 - E\,f^2\,|\varphi|^2\Big)\,\mathrm{d}s\,\mathrm{d}t$$

Furthermore, we find easily $\int_{\mathbb{R}} (\partial_s\bar\varphi\,\varphi + \bar\varphi\,\partial_s\varphi)\,\mathrm{d}s = 0$ and $-\Delta_D^\omega f - \beta_0^2\,\partial_\alpha^2 f - E\,f = 0$; it allows us to conclude the proof,

$$Q_0[\psi] - E\,\|\psi\|^2 = \int_{\mathbb{R}\times\omega} f^2\,\big(|\nabla_t\varphi|^2 + |\partial_s\varphi + \beta_0\,\varphi'_\tau|^2\big)\,\mathrm{d}s\,\mathrm{d}t\ \geq 0$$

Local twist perturbations

Look now what happens if the translation invariance of the tube is broken, velocity of the twisting being given by

$$\dot{\theta}(s) = \beta_0 - \beta(s)\,,$$

where $\beta(\cdot)$ is bounded, $\operatorname{supp}\beta \subset [-s_0, s_0]$ for some $s_0 > 0$

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Put $\Omega_\beta := \mathcal{L}(\mathbb{R} \times \omega)$, and let H_β on $L^2(\Omega_\beta)$ be associated with

$$Q_\beta[\psi] := \int_{\Omega_\beta} |\nabla\psi|^2 \,\mathrm{d}s\,\mathrm{d}t$$

defined on $D(Q_\beta) = \mathcal{H}_0^1(\Omega_\beta)$. Since $\operatorname{supp}\beta$ is compact by assumption, it is straightforward to check that

$$\sigma_{ess}(H_\beta) = \sigma_{ess}(H_0) = [E, \infty)$$

Eigenvalues by a slowed-down twist

Theorem [E.-Kovařík'05]: Assume that ω is not rotationally symmetric and that

$$\int_{-s_0}^{s_0} (\dot{\theta}^2(s) - \beta_0^2)\, \mathrm{d}s < 0\,,$$

then H_β has at least one eigenvalue of finite multiplicity below the threshold of the essential spectrum

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Proof: We will construct a trial function from the threshold resonance corresponding to the bottom of the essential spectrum. Given $\delta > 0$ we put $\Psi_\delta(s,t) = f(t)\,\varphi(s)$, where

$$\varphi(s) = \begin{cases} e^{\delta\,(s_0+s)} & \text{if} \quad s \le -s_0 \,, \\ 1 & \text{if} \quad -s_0 \le s \le s_0 \,, \\ e^{-\delta\,(s-s_0)} & \text{if} \quad s \ge s_0 \,. \end{cases}$$

Proof, continued

Obviously $\Psi_\delta \in D(Q_\beta)$. By a direct calculation we find

$$Q_\beta[\Psi_\delta] - E\,\|\Psi_\delta\|^2 = \delta\,\|f\|^2_{L^2(\omega)} + \|f'_\tau\|^2_{L^2(\omega)} \int_{-s_0}^{s_0} (\dot\theta^2(s) - \beta_0^2)\,\mathrm{d}s$$

and furthermore, $\|\Psi_\delta\|^2 = (\delta^{-1} + 2s_0)\,\|f\|^2_{L^2(\omega)}$

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Consequently, in the limit $\delta \to 0$ we get

$$\frac{Q_\beta[\Psi_\delta] - E\,\|\Psi_\delta\|^2}{\|\Psi_\delta\|^2} = \delta\,\frac{\|f'_\tau\|^2_{L^2(\omega)}}{\|f\|^2_{L^2(\omega)}} \int_{-s_0}^{s_0} (\dot\theta^2(s) - \beta_0^2)\,\mathrm{d}s + \mathcal{O}(\delta^2)$$

By the lemma $\|f'_\tau\|^2_{L^2(\omega)} > 0$ so the l.h.s. of the last relation is negative for δ small enough. $\square$

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The result can be extended to the critical case under somewhat stronger assumption on the regularity of $\dot{\theta}$. We also have to suppose that the twisting is “not fully reverted” by the perturbation.

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The result can be extended to the critical case under somewhat stronger assumption on the regularity of $\dot\theta$. We also have to suppose that the twisting is “not fully reverted” by the perturbation.

Theorem [E.-Kovařík’05]: Assume that ω is not rotationally symmetric. In addition, let $\dot\theta(s) + \beta_0 > 0$ hold for $|s| \le s_0$, and moreover, let $\ddot\theta$ exist being of the class $L^2([-s_0, s_0])$. If

$$\int_{-s_0}^{s_0} \left(\dot\theta^2(s) - \beta_0^2\right) \mathrm{d}s = 0,$$

the operator H_β has at least one eigenvalue of finite multiplicity below the threshold of the essential spectrum.

The critical case, proof

Proof: We use the *Goldstone-Jaffe trick*, improving the trial function by a deformation in the central region,

$$\Psi_{\delta,\gamma}(s,t) := f(t)\,\varphi_\gamma(s)\,,$$

where for a fixed $\gamma > 0$ we put

$$\varphi_\gamma(s) = \begin{cases} e^{\delta\,(s_0+s)} & \text{if} \quad s \le -s_0\,, \\ 1+\gamma\,(\beta_0 - \dot\theta(s)) & \text{if} \quad -s_0 \le s \le s_0\,, \\ e^{-\delta\,(s-s_0)} & \text{if} \quad s \ge s_0\,. \end{cases}$$

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Similarly as in the previous proof we check that

$$Q_\beta[\Psi_{\delta,\gamma}] - E\,\|\Psi_{\delta,\gamma}\|^2 = \int_{\mathbb{R}\times\omega} \left(\varphi_\gamma^2\,(f_\tau')^2 \left(\dot\theta^2(s) - \beta_0^2\right) + f^2\,(\varphi_\gamma')^2\right) \mathrm{d}s\,\mathrm{d}t$$

Proof, continued

Under given assumptions we get as $\gamma,\,\delta \to 0$

$$\int_{-s_0}^{s_0} \varphi_\gamma^2 \left(\dot\theta^2(s) - \beta_0^2\right) \mathrm{d}s = -2\gamma \int_{-s_0}^{s_0} \left(\dot\theta(s) - \beta_0\right)^2 \left(\dot\theta(s) + \beta_0\right) \mathrm{d}s + \mathcal{O}(\gamma^2)$$

and

$$\int_{\mathbb{R}} (\varphi_\gamma')^2 \, \mathrm{d}s = \delta + \gamma^2 \int_{-s_0}^{s_0} \left(\ddot\theta(s)\right)^2 \mathrm{d}s = \mathcal{O}(\gamma^2) + \mathcal{O}(\delta)$$

Proof, continued

Under given assumptions we get as $\gamma,\,\delta\to 0$

$$\int_{-s_0}^{s_0}\varphi_\gamma^2\,\left(\dot\theta^2(s)-\beta_0^2\right)\,\mathrm{d}s=-2\gamma\int_{-s_0}^{s_0}\left(\dot\theta(s)-\beta_0\right)^2\left(\dot\theta(s)+\beta_0\right)\,\mathrm{d}s+\mathcal{O}(\gamma^2)$$

and

$$\int_{\mathbb{R}}(\varphi_\gamma')^2\,\mathrm{d}s=\delta+\gamma^2\int_{-s_0}^{s_0}\left(\ddot\theta(s)\right)^2\,\mathrm{d}s=\mathcal{O}(\gamma^2)+\mathcal{O}(\delta)$$

Combining the last two equations we then get

$$\frac{Q_\beta[\Psi_{\delta,\gamma}]-E\,\|\Psi_{\delta,\gamma}\|^2}{\|\Psi_{\delta,\gamma}\|^2}\;=\;-2\,\gamma\,\delta\,\frac{\|f_\tau'\|^2_{L^2(\omega)}}{\|f\|^2_{L^2(\omega)}}\int_{-s_0}^{s_0}\left(\dot\theta(s)-\beta_0\right)^2\left(\dot\theta(s)+\beta_0\right)\,\mathrm{d}s$$
$$+\delta\,\mathcal{O}(\gamma^2)+\mathcal{O}(\delta^2)\,.$$

Setting $\gamma=\sqrt{\delta}$ we have then to take δ small enough $\quad\square$

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- Also, under which conditions it has *gaps?* . . . *etc.*

The talk was based on

[EKK05] T. Ekholm, H. Kovařík, D. Krejčiřík: A Hardy inequality in twisted waveguides, `math-ph/0512050`

[EK05] P.E., H. Kovařík: Spectrum of the Schrödinger operator in a perturbed periodically twisted tube, *Lett. Math. Phys.* **73** (2005), 183–192.

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It remains to say:

Happy birthday, Michael!