Geometrically induced bound states in Dirichlet layers

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Talk overview

- Physical and mathematical motivation
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- Preliminaries: geometry of a curved layer
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- Weak coupling: mildly curved layers
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- Weak coupling: mildly curved layers
- Some open questions
Motivation

Problem: properties of a quantum particle confined to a *curved layer* of fixed width built over a surface

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- See [Jensen-Koppe ’71], [Tolar ’78], [da Costa ’81], ...

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- We are interested primarily in relations between \textit{geometry} and \textit{spectral properties}, i.e. a trademark topic of mathematical physics
Motivation: semiconductor films

A natural model for *dilute electron gas* in *semiconductor films* built on a *curved substrate*. Recall that a typical mesoscopic system has

- **small size**: submicron, down to nanometers
- **high purity**: mean free path $\gg$ system size
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One typically one assumes *hard wall (Dirichlet)* boundary conditions. It is an idealization, in reality rather a finite potential jump.
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A lot is known about QM in strips or tubes modelling *quantum wires*. Recall some results:

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- Thin enough bent waveguides have resonances
- Thin enough periodically curved waveguides have open gaps, etc.
The surface $\Sigma$ in $\mathbb{R}^3$ supposed to be $C^2$-smooth and to have at least one pole (i.e., exponential mapping $\exp_0: T_0 \Sigma \to \Sigma$ is a diffeomorphism). Hence $\sigma$ is diffeomorphic to $\mathbb{R}^2$, i.e. simply connected and non-compact. Using geodesic polar coordinates we parametrize

$$p: \Sigma_0 \to \mathbb{R}^3 : \{q := (s, \theta) \mapsto p(q) \in \Sigma\}, \quad \Sigma_0 := (0, \infty) \times S^1$$
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The tangent vectors $p, \mu := \partial p/\partial q^\mu$ are linearly independent and their cross-product defines a unit normal field $n$ on $\Sigma$. 
Preliminaries

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The layer $\Omega := \mathcal{L}(\Omega_0)$ of width $d = 2a$ over $\Sigma$, where $\Omega_0 := \Sigma_0 \times (-a, a)$, is defined by the map

$$\mathcal{L} : \Omega_0 \to \mathbb{R}^3 : \{(q, u) \mapsto \mathcal{L}(q, u) := p(q) + u n(q) \in \Omega\}$$
Motivation: surfaces with poles

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The assumption is nontrivial. *Example* [Gromol-Meyer ’69]:

![Diagram](spherical_elliptical_legs.png)

However, the assumption is not necessary for the spectral result we are going to derive. Later we get rid of it.
The *surface metric* in the geodesic polar coordinates is diagonal, \((g_{\mu\nu}) = \text{diag} (1, r^2)\), where \(r^2 \equiv g := \det(g_{\mu\nu})\) is the squared Jacobian of the exponential mapping which satisfies *Jacobi equation*

\[
\ddot{r}(s, \vartheta) + K(s, \vartheta) r(s, \vartheta) = 0, \quad r(0, \vartheta) = 0, \quad \dot{r}(0, \vartheta) = 1
\]

Integrating it we get \(\int_0^\infty r(s, \theta) \, d\theta \leq C s\) for some \(C > 0\) provided the total curvature \(K\) defined below is finite.
Preliminaries: surface geometry

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In addition to \( g_{\mu\nu} := p_{,\mu} \cdot p_{,\nu} \) we introduce second fundamental form \( h_{\mu\nu} := -n_{,\mu} \cdot p_{,\nu} \) with \( h := \det(h_{\mu\nu}) \) and Weingärten map \( h^\mu_{\nu} := g^{\mu\rho} h_{\rho\nu} \) which determine

- **Gauss curvature** \( K := \det(h^\mu_{\nu}) = h/g \)
- **mean curvature** \( M := \frac{1}{2} \text{Tr}(h^\mu_{\nu}) = \frac{1}{2} g^{\mu\nu} h_{\mu\nu} \)
Preliminaries: total curvatures

Using *invariant surface element*, \( d\Sigma := g^{1/2} d^2 q \equiv g^{1/2} dq^1 dq^2 \), we introduce global quantities, in particular, *total curvatures*

\[
\mathcal{K} := \int_{\Sigma} K d\Sigma \quad \text{and} \quad \mathcal{M}^2 := \int_{\Sigma} M^2 d\Sigma ;
\]

we will suppose that the first one is finite, \( K \in L^1(\Sigma_0, d\Sigma) \).
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For a compact manifold \( G \) with a smooth boundary we have

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\mathcal{K}_G + \oint_{\partial G} k_g ds = 2\pi \quad \text{by Gauss-Bonnet theorem}
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For a compact manifold \( \mathcal{G} \) with a smooth boundary we have

\[
\mathcal{K}_\mathcal{G} + \oint_{\partial \mathcal{G}} k_g ds = 2\pi \quad \text{by *Gauss-Bonnet theorem*}
\]

In particular, if \( \Sigma \) is a *locally deformed plane* we choose \( \partial \mathcal{G} \)
outside the deformation, so \( \mathcal{K}_\mathcal{G} = \mathcal{K}_\Sigma = 0 \)
**Preliminaries: layer geometry**

*Metric tensor*, \( G_{ij} := \mathcal{L}_i \cdot \mathcal{L}_j \), of the layer (regarded as a manifold with boundary in \( \mathbb{R}^3 \)) has the block form

\[
(G_{ij}) = \begin{pmatrix}
(G_{\mu\nu}) & 0 \\
0 & 1
\end{pmatrix}
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with \( G_{\nu\mu} = (\delta^\sigma_\nu - uh^\sigma_\nu)(\delta^\rho_\tau - uh^\rho_\tau)g_{\rho\mu} \).
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Recall that the ev’s of Weingärten map matrix are *principal curvatures* \( k_1, k_2 \), and that \( K = k_1 k_2, \ M = \frac{1}{2}(k_1 + k_2) \)
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Then we can express the determinant, $G := \det(G_{ij})$ as

$$G = g [(1 - uk_1)(1 - uk_2)]^2 = g(1 - 2Mu + Ku^2)^2$$

In particular, the *volume element* is $d\Omega := G^{1/2} d^2q \, du$.
Preliminaries: assumptions

For the moment we adopt the following hypotheses:

\begin{align*}
\langle \Sigma_0 \rangle & \quad K \in L^1(\Sigma_0, d\Sigma) \\
\langle \Omega_0 \rangle & \quad \Omega \text{ is not self-intersecting, i.e. } L \text{ is injective} \\
\langle \Omega_1 \rangle & \quad a < \rho_m := (\max \{\|k_1\|_{\infty}, \|k_2\|_{\infty}\})^{-1}
\end{align*}
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\end{align*}

The last one ensures that \( \mathcal{L} \) is a diffeomorphism, in particular, that \( \Omega \) has a smooth boundary. Furthermore, \( \langle \Omega 1 \rangle \) also implies a useful estimate,

\[ C_- g_{\mu \nu} \leq G_{\mu \nu} \leq C_+ g_{\mu \nu} \quad \text{with} \quad 0 < C_- < 1 < C_+ < 4 \]

and the constants expressed in terms of the \textit{minimal normal curvature radius} \( \rho_m \) as \( C_\pm := \left( 1 \pm a \rho_m^{-1} \right)^2 \).
Neglecting physical constants the Hamiltonian is identified with the Dirichlet Laplacian $-\Delta_D^\Omega$ on $L^2(\Omega)$ with the usual properties, e.g., the form domain is $W^{1,2}_0(\Omega)$. 
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In the coordinates $(q, u)$ it acquires Laplace-Beltrami form

$$H := -G^{-1/2} \partial_i G^{1/2} G^{ij} \partial_j$$
on $L^2(\Omega_0, G^{1/2} d^2 q \, d u)$,

or $H = U(-\Delta^\Omega_D)U^{-1}$ with unitary $U : L^2(\Omega) \to L^2(\Omega_0, d\Omega)$. 
Hamiltonian: curvilinear coordinates

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If $\Sigma$ is not $C^3$-smooth, $H$ is understood in the form sense

$$Q(\psi) := \|H^{1/2}\psi\|_G^2 = (\psi, i, G^{ij} \psi, j)_G, \quad D(Q) = W^{1,2}_0(\Omega_0, d\Omega),$$

where "$G$" indicates the norm and the inner product in the above Hilbert space.
Hamiltonian: decomposition

The block form of $G_{ij}$ yields $H = H_1 + H_2$ with

$$H_1 := -G^{-1/2} \partial_{\mu} G^{1/2} G^{\mu\nu} \partial_{\nu} = -\partial_{\mu} G^{\mu\nu} \partial_{\nu} - 2 F_{,\mu} G^{\mu\nu} \partial_{\nu},$$

$$H_2 := -G^{-1/2} \partial_{3} G^{1/2} \partial_{3} = -\partial_{3}^2 - 2 \frac{K u - M}{1 - 2 M u + K u^2} \partial_{3},$$

where $F := \ln G^{1/4}$ and $F_{,3}$ is given explicitly in $H_2$. 

Wrocław University, Institute of Theoretical Physics, April 16, 2004 – p.13/50
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where $F := \ln G^{1/4}$ and $F,_{3}$ is given explicitly in $H_2$

An alternative form, with the factor $1 - 2Mu + Ku^2$ removed from the weight $G^{1/2}$, is obtained by another unitary transformation $\hat{U} : L^2(\Omega_0, d\Omega) \rightarrow L^2(\Omega_0, d\Sigma du)$,

$$\psi \mapsto \hat{U}\psi := (1 - 2Mu + Ku^2)^{1/2}\psi,$$

giving $\hat{H} := \hat{U} H \hat{U}^{-1}$. The norm in the corresponding Hilbert space is indicated by the subscript “g”
Hamiltonian: decomposition

The operator $\hat{H}$ contains an effective potential; introducing $J := \frac{1}{2} \ln(1 - 2Mu + Ku^2)$ we rewrite it as follows,

$$\hat{H} = -g^{-1/2} \partial_i g^{1/2} G^{ij} \partial_j + V, \quad V = g^{-1/2} (g^{1/2} G^{ij} J_{,j})_{,i} + J_{,i} G^{ij} J_{,j}$$
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This yields $\hat{H} = \hat{H}_1 + \hat{H}_2$, where $\hat{H}_1$ has the above form with summation over Greek indices and

$$\hat{H}_2 = -\partial_3^2 + V_2, \quad V_2 = \frac{K - M^2}{(1 - 2Mu + Ku^2)^2}$$
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In analogy with the curved tube case it is illustrative to write $\hat{H} = \hat{H}_q - \partial_3^2$, where $\hat{H}_q := \hat{H}_1 + V_2$
Heuristic considerations

In thin layers, $a \ll \rho_m$, the longitudinal and transverse variables are \textit{asymptotically decoupled}, because

$$H_q := -g^{-1/2} \partial_\mu g^{1/2} g^{\mu \nu} \partial_\nu + K - M^2 + \mathcal{O}(a\rho_m^{-1}) ;$$

notice that in distinction from the tube case the surface cannot be fully “ironed”, the surface geometry persists
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notice that in distinction from the tube case the surface cannot be fully “ironed”, the surface geometry persists.

The additional potential $K - M^2$ rewrites in terms of principal curvatures as $-\frac{1}{4}(k_1 - k_2)^2$. It is attractive unless

- $\Sigma$ is planar, $k_1 = k_2 = 0$
- $\Sigma$ is spherical, $k_1 = k_2$, however, a noncompact $\Sigma$ clearly cannot be spherical globally
Examples of the effective interaction

Effective Potential \[ V_{\text{eff}} = -\frac{1}{4} (k_+ - k_-)^2 \]

Paraboloid of Revolution \[ z = x^2 + y^2 \]

Hyperbolic Paraboloid \[ z = x^2 - y^2 \]

Monkey Saddle \[ z = x^3 - 3xy^2 \]

The minima of \( V_{\text{eff}} \) are marked by the dark red colour.
Essential spectrum threshold

**Notation:** we use eigenfunctions \( \{ \chi_n \}_{n=1}^{\infty} \) of the transverse operator \((-\partial^2_D)_D\) given by \( \sqrt{\frac{2}{d}} \left( \cos \frac{\kappa_n}{\sin u} \right) \) for \( n \) odd, where \( \kappa_n^2 := (\kappa_1 n)^2 \) with \( \kappa_1 := \pi/d \) are the corresponding ev’s
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**One more assumption:** \( \Sigma \) is *asymptotically planar*, i.e.

\[
\langle \Sigma 0 \rangle \quad K, M \to 0 \quad \text{holds as} \quad s \to \infty
\]
**Essential spectrum threshold**

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\[ \langle \Sigma 0 \rangle \]  

\( K, M \to 0 \) holds as \( s \to \infty \)

**Theorem** [Duclos-E.-Krejčiřík, 2001]: Assume \( \langle \Omega 0 \rangle, \langle \Omega 1 \rangle \) and \( \langle \Sigma 0 \rangle \), then we have

\[ \inf \sigma_{ess}(-\Delta_{\Omega D}) \geq \kappa_1^2 \]
Divide $\Omega$ into an exterior and interior by extra *Neumann b.c.* at $s = s_0$, then $H \geq H_N^{\text{int}} \oplus H_N^{\text{ext}}$. The interior does not contribute to $\sigma_{\text{ess}}$, so by minimax principle we infer

$$\inf \sigma_{\text{ess}}(H) \geq \inf \sigma_{\text{ess}}(H_N^{\text{ext}}) \geq \inf \sigma(H_N^{\text{ext}})$$

In the exterior we have for all $\psi \in D(Q_N^{\text{ext}})$ the estimate

$$Q_N^{\text{ext}}(\psi) \geq \|\psi,3\|_{G,\text{ext}}^2 \geq \inf_{\Omega_{\text{ext}}} \{1 - 2Mu + Ku^2\} \|\psi,3\|_{g,\text{ext}}^2$$

$$\geq \left(1 - \sup_{\Sigma_{\text{ext}}}\{2a|M| + a^2|K|\}\right) \kappa_1^2 \|\psi\|_{g,\text{ext}}^2$$

$$\geq \frac{1 - \sup_{\Sigma_{\text{ext}}}\{2a|M| + a^2|K|\}}{1 - \inf_{\Sigma_{\text{ext}}}\{2a|M| + a^2|K|\}} \kappa_1^2 \|\psi\|_{G,\text{ext}}^2$$

$$= (1 + o(s_0)) \kappa_1^2 \|\psi\|_{G,\text{ext}}^2 \quad \Box$$
Curvature-induced binding, $\mathcal{K} \leq 0$

**Theorem [Duclos-E.-Krejčičík, 2001]**: Assume $\langle \Omega_0 \rangle$, $\langle \Omega_1 \rangle$ and $\langle \Sigma_1 \rangle$, and suppose that $\Sigma$ is not planar. If $\mathcal{K} \leq 0$, then

$$\inf \sigma(-\Delta_{D}) < \kappa_1^2$$

In particular, $\sigma_{\text{disc}}(-\Delta_{D}) \neq \emptyset$ if $\langle \Sigma_0 \rangle$ holds.
Curvature-induced binding, $\mathcal{K} \leq 0$

**Theorem** [Duclos-E.-Krejčiřík, 2001]: Assume $\langle \Omega_0 \rangle$, $\langle \Omega_1 \rangle$ and $\langle \Sigma_1 \rangle$, and suppose that $\Sigma$ is not planar. If $\mathcal{K} \leq 0$, then

$$\inf \sigma(-\Delta_{\mathcal{D}}^\mathcal{O}) < \kappa_1^2$$

In particular, $\sigma_{\text{disc}}(-\Delta_{\mathcal{D}}^\mathcal{O}) \neq \emptyset$ if $\langle \Sigma_0 \rangle$ holds.

**Sketch of the proof:** By a variational argument, seeking a trial function $\Psi$ from $\mathcal{Q}(H)$ such that

$$\tilde{\mathcal{Q}}(\Psi) := \mathcal{Q}(\Psi) - \kappa_1^2 \|\Psi\|_G^2 < 0$$

It is convenient to split the Hamiltonian form, $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ with parts associated to $H_1$ and $H_2$ introduced above. We employ *Goldstone-Jaffe trick*, choosing radially symmetric $\psi(s, \vartheta, u) := \varphi(s) \chi_1(u)$ with $\varphi$ to be specified.
$\mathcal{K} \leq 0$, sketch of the proof

Using the factorized form of $\psi$ we get directly

$$Q_2(\psi) - \kappa_1^2 \|\psi\|_G^2 = (\psi, K\psi)_g$$

On the other hand, the “longitudinal kinetic part” $Q_1(\psi)$ can be estimated by the radial gradient norm of $\psi$ as

$$Q_1(\psi) \leq C_1 \int_0^{\infty} |\varphi(s)|^2 s \, ds$$

with some $C_1 > 0$. To make it small we need a suitable family of radial functions such that $\psi \in \mathcal{Q}(H)$; we choose them as scaled Macdonald functions outside a circle, i.e.

$$\varphi_\sigma(s) := \min \left\{ 1, \frac{K_0(\sigma s)}{K_0(\sigma s_0)} \right\}$$
\( \mathcal{K} \leq 0 \), sketch of the proof

It is straightforward to compute the integral; we get

\[
\exists C_2 > 0 : \quad \int_0^\infty |\dot{\phi}_\sigma(s)|^2 s \, ds < \frac{C_2}{|\ln \sigma s_0|},
\]

and therefore \( Q_1(\psi_\sigma) \to 0^+ \) as \( \sigma \to 0^+ \). We assume \( \langle \Sigma 1 \rangle \), so by dominated the first part of the shifted energy form tends to \( \mathcal{K} \) as \( \sigma \to 0^+ \); this proves the theorem if \( \mathcal{K} < 0 \).
\[ \mathcal{K} \leq 0, \text{ sketch of the proof} \]

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If \( \mathcal{K} = 0 \) we follow GJ idea choosing \( \Psi_{\sigma,\epsilon} := \psi_\sigma + \epsilon \Theta \), where \( \Theta(q, u) := j(q)^2 u \chi_1(u) \) with \( j \in C_0^\infty((0, s_0) \times S^1) \); it gives

\[ \tilde{Q}(\Psi_{\sigma,\epsilon}) = \tilde{Q}(\psi_\sigma) + 2\epsilon \tilde{Q}(\Theta, \psi_\sigma) + \epsilon^2 \tilde{Q}(\Theta) \]

Since \( \tilde{Q}(\Theta, \psi_\sigma) = -\frac{1}{d} (j, M)_g \neq 0 \) in general, the sum of the last two terms can be made negative; then \( \tilde{Q}(\Psi_{\sigma,\epsilon}) < 0 \) will hold for \( \sigma \) small enough. \( \square \)
\( \mathcal{K} \leq 0 \), examples

The theorem applies to layers built over Cartan-Hadamard surfaces, i.e. geodesically complete simply connected non-compact ones with \( \mathcal{K} \leq 0 \) (then each point is a pole)

Locally curved plane has \( \mathcal{K} = 0 \) by Gauss-Bonnet, the same is true for surfaces with curvatures which are not compactly supported but decay fast enough
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- **Hyperbolic paraboloid**: the simple quadric given in \( \mathbb{R}^3 \) by the equation \( z = x^2 - y^2 \) is an asymptotically planar surface with \( \mathcal{K} = -2\pi \)
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- **Hyperbolic paraboloid**: the simple quadric given in $\mathbb{R}^3$ by the equation $z = x^2 - y^2$ is an asymptotically planar surface with $\mathcal{K} = -2\pi$

- **Monkey saddle**: another example of a saddle surface is $z = x^3 - 3xy^2$; it satisfies again $\langle \Sigma 1 \rangle$ and $\mathcal{K} = -4\pi$
Other sufficient conditions

The GJ trick – constructing a trial function starting from a factorized function $\psi(s, v, u) := \varphi(s) \chi_1(u)$ – does not work for $\mathcal{K} > 0$. However, other sufficient conditions can still be obtained variationally:
Other sufficient conditions

The GJ trick – constructing a trial function starting from a factorized function $\psi(s, \vartheta, u) := \varphi(s) \chi_1(u)$ – does not work for $K > 0$. However, other sufficient conditions can still be obtained variationally:

**Theorem** [Duclos-E.-Krejčiřík, 2001]: Assume $\langle \Omega_0 \rangle$ and $\langle \Omega_1 \rangle$ and suppose that $\Sigma$ is $C^3$-smooth and non-planar. In addition, let one of the following conditions be valid:

- the layer $\Omega$ is *thin enough*
- we have $\langle \Sigma 1 \rangle$, $M = \infty$, and
  $$\langle \Sigma 2 \rangle$$ the covariant derivative $\nabla_g M \in L^2(\Sigma_0, d\Sigma)$

Then $\inf \sigma(-\Delta^{\Omega_D}) < \kappa_1^2$, in particular, curvature-induced bound states exist under the assumption $\langle \Sigma 0 \rangle$
Sketch of the proof

Trial function \( \Psi_\sigma(s, \vartheta, u) := (1 + M(s, \vartheta)u) \psi_\sigma(s, u) \) gives

\[
Q_1(\Psi_\sigma) \leq 2(C_+/C_-)^2 \left( (1 + a\|M\|_\infty)^2 \|\dot{\psi}_\sigma\|_g^2 + a^2\|\psi_\sigma \nabla_g M\|_g^2 \right)
\]
small as \( \sigma \to 0 \)

\[
O(a^2)
\]

\[
Q_2(\Psi_\sigma) - \kappa_1^2\|\psi\|_G^2 = (\psi_\sigma, (K - M^2)\psi_\sigma)_g + \frac{\pi^2 - 6}{12\kappa_1^2} (\psi_\sigma, KM^2\psi_\sigma)_g
\]

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If $a$ is small enough, choosing small $\sigma$ we can achieve that the sum dominated by $(\psi_\sigma, (K - M^2)\psi_\sigma)_g < 0$
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Under the second assumption, $(\psi_\sigma, -M^2\psi_\sigma)_g \to -\infty$ as $\sigma \to 0^+$, while the other terms remain finite. $\square$
Cylindrically symmetric layers

Another sufficient condition can be derived for layers invariant w.r.t. rotations around a fixed axis in $\mathbb{R}^3$ with $\Sigma$ parameterized by means of $r, z \in C^2 ((0, \infty))$ as

$$p : \Sigma_0 \to \mathbb{R}^3 : \{(s, \vartheta) \mapsto (r(s) \cos \vartheta, r(s) \sin \vartheta, z(s))\}$$

It is a geodesic polar coordinate chart if we require

$$\dot{r}^2 + \dot{z}^2 = 1 \; ; \; \text{then also} \; \dot{r}\ddot{r} + \dot{z}\ddot{z} = 0$$

The Weingärten tensor is $(h^\nu_\mu) = \text{diag} (k_s, k_\vartheta)$ with the principal curvatures $k_s = \dot{r}\ddot{z} - \dddot{r}z$ and $k_\vartheta = \dddot{z} r$. We have

$$\mathcal{K} + 2\pi \dot{r}(\infty) = 2\pi \; , \; \text{where} \; \dot{r}(\infty) := \lim_{s \to \infty} \dot{r}(s)$$

by Gauss-Bonnet theorem, and since $0 \leq \dot{r}(\infty) \leq 1$, such a cylindrically invariant surface $\Sigma$ always has $0 \leq \mathcal{K} \leq 2\pi$
Cylindrically symmetric layers

We exclude the case already resolved and assume $K > 0$, i.e. $0 \leq \dot{r}(\infty) < 1$. Using the above parametrization we get

**Lemma:** Let $K > 0$, then there are $\delta > 0$ and $s_0 > 0$ s.t.

$$\forall s \geq s_0 : \quad \frac{\delta}{r(s)} \leq |k_\vartheta(s)| \leq \frac{1}{r(s)}$$

and $k_\vartheta(s)$ does not change sign. It follows that $k_\vartheta$ is not integrable in $L^1(\mathbb{R}_+)$. If $\langle \Sigma 1 \rangle$ is satisfied, we have $M = \infty$
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**Theorem** [Duclos-E.-Krejčiřík, 2001]: Assume $\langle \Omega 0 \rangle$, $\langle \Omega 1 \rangle$ and $\langle \Sigma 1 \rangle$, and suppose that $\Sigma$ is a surface of revolution with $\mathcal{K} > 0$. Then $\inf \sigma(-\Delta^D_\Omega) < \kappa_1^2$, in particular, $\sigma_{\text{disc}}(d - \Delta^D_\Omega) \neq \emptyset$ holds under the assumption $\langle \Sigma 0 \rangle$
Sketch of the proof

By assumption $M$ dominates over $K$ in effective potential at large distances, hence we choose trial functions supported there. Consider sequences $\{n^i\}_{n=1}^\infty$, $i = 1, 2, 3$, and put

$$\varphi_n(s) := \frac{\ln(sn^{-i})}{\ln(n^{-i})}, \quad \phi_n(s) := \frac{\varphi_n(s)}{s}, \quad (i, j) \in \{(1, 2), (3, 2)\}$$

if $\min\{n^i, n^j\} < s \leq \max\{n^i, n^j\}$ and zero otherwise. We employ functions $\Psi_{n,\varepsilon}(s, u) := (\varphi_n(s) + \varepsilon\phi_n(s)u)\chi_1(u)$ which belong to form domain of $H$ and are uniformly bounded.
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By a direct computation and simple estimates we get

$$\lim_{n \to \infty} \tilde{Q}[\Psi_{n, \varepsilon}] = \lim_{n \to \infty} \left[\varepsilon^2 \|\phi_n\|_\Sigma^2 - 2\varepsilon(\varphi_n, M\phi_n)_\Sigma\right]$$

if the r.h.s. limit exists, where the norms refer to $L^2(\Sigma, d\Sigma_0)$. 
Sketch of the proof

We choose \( \varepsilon \equiv \varepsilon_n := (\varphi_n, M\phi_n)^{-1} \) which makes sense as the integral diverges; thus one has to compare \(-2\) with

\[
\lim_{n \to \infty} \frac{(\phi_n, \phi_n)^{\Sigma}}{(\varphi_n, M\phi_n)^{2\Sigma}}
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Sketch of the proof

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\]

Now finally we use rotational symmetry. Since \( k_s \in L^1(\mathbb{R}_+) \) and \( \phi_n \) is chosen to eliminate the weight \( r \), the meridian curvature does not contribute in the denominator, while in view of the lemma \( k_{\psi r} \) behaves as one at infinity. Consequently, the limit in question is

\[
\int_0^\infty \frac{\phi_n(s)^2 s ds}{(\int_0^\infty \varphi_n(s)\phi_n(s) ds)^2} = \frac{1}{\int_0^\infty \phi_n(s)^2 s ds} = \frac{3}{\ln(n^2)} \to 0,
\]

and thus \( \lim_{n \to \infty} \tilde{Q}(\Psi_n, \varepsilon) \to -2 \) as we sought to prove \( \square \)
Remarks

\textit{Partial wave decomposition:} one can decompose $-\Delta^\Omega_D$ to angular momentum subspaces and employ 2D methods. It is not much simpler, but one gets an insight: the trial function could be supported in the far off region where the \textit{centrifugal term is weak}.
Remarks

- **Partial wave decomposition**: one can decompose $-\Delta_{\mathcal{D}}^\Omega$ to angular momentum subspaces and employ 2D methods. It is not much simpler, but one gets an insight: the trial function could be supported in the far off region where the *centrifugal term is weak*.

- **Layers without bound states**: if you “close” $\Sigma$ too much the discrete spectrum may be lost. *Example*: let $\Sigma$ be a cylinder with a hemispherical “cap”, then by Neumann bracketing we check that $\sigma_{\text{disc}}(-\Delta_{\mathcal{D}}^\Omega) = \emptyset$. While it does not satisfy our smoothness assumptions, a counterexample is obtained using domain continuity. The reason is, of course, that such a $\Sigma$ *ceases to be asymptotically planar* pushing $\inf \sigma_{\text{ess}}(-\Delta_{\mathcal{D}}^\Omega)$ down.
Generalizations

Let $\Omega$ be built over $\Sigma$ which is complete non-compact connected $C^2$-smooth surface, and suppose that $\langle \Omega 0 \rangle$, $\langle \Omega 1 \rangle$ and $\langle \Sigma 1 \rangle$ are satisfied.

- Under $\langle \Sigma 0 \rangle$ we have $\inf \sigma_{\text{ess}} \left( -\Delta_D^{\Omega} \right) = \kappa_1^2$
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- More important, we have **new sufficient conditions:** \( \inf \sigma (-\Delta_{\Omega_D}) < \kappa_1^2 \) holds if \( \Sigma \) contains a **cylindrically symmetric end** with a **positive total Gauss curvature**, and

- the same is true if the generating surface \( \Sigma \) **is not conformally equivalent to the plane**
inf \sigma_{\text{ess}} \left( -\Delta_{\Omega}^{D} \right) \text{ revisited}

The lower bound by \kappa_{1}^{2} can be proved under the more general assumptions; the argument based on Neumann bracketing generalizes easily.
inf \sigma_{ess} \left( -\Delta_D \right) \text{ revisited}

The lower bound by $\kappa_1^2$ can be proved under the more general assumptions; the argument based on Neumann bracketing generalizes easily.

The upper bound: If $K \to 0$ at infinity, to any $\varepsilon > 0$ there is an infinite-dimensional $D_g \subset C_0^\infty(\Sigma)$ s.t. $\|\nabla g \varphi\|_g \leq \varepsilon \|\varphi\|_g$ holds for $\varphi \in D_g$. Then we employ the identity

$$\|\nabla \varphi \chi_1\|^2 = \|\nabla \varphi \chi_1\|^2 - (\varphi \chi_1, \varphi \Delta \chi_1)$$

The first term is estimated by $(C_+/C_-^2) \varepsilon^2 \|\varphi \chi_1\|^2$, while the one can be rewritten as

$$-(\varphi \Delta \chi_1, \varphi \chi_1) = \kappa_1^2 \|\varphi \chi_1\|^2 + (\varphi \chi_1', 2M_u \varphi \chi_1),$$

where $M_u := \frac{M-Ku}{1-2Mu+Ku^2}$ refers to “parallel” surface $L(\Sigma \times \{u\})$. 

Wroclaw University, Institute of Theoretical Physics, April 16, 2004 – p.31/50
Inf \sigma_{\text{ess}} \left( -\Delta_D^\Theta \right) \text{ revisited}

Integrating the last term by parts in \( u \) we conclude that for any \( \varepsilon > 0 \) there is \( \mathcal{D} := \mathcal{D}_g \otimes \{\chi_1\} \subset C^\infty_0(\Omega) \) such that

\[
\forall \psi \in \mathcal{D} : \quad \| \nabla \psi \|^2 - (\psi, K_u \psi) \leq (\kappa_1^2 + (C_+/C_-^2) \varepsilon^2) \| \psi \|^2,
\]

where \( K_u := \frac{K}{1-2Mu+Ku^2} \) is the Gauss curvature of the above indicated parallel surface
\[ \inf \sigma_{\text{ess}} \left( -\Delta_D^\Omega \right) \text{ revisited} \]

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\[
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\]

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This proves \( \inf \sigma_{\text{ess}}(-\Delta_D^\Omega - K_u) \leq \kappa_1^2 \). Since \( K_u \) vanishes at infinity by assumption, the operator \( K_u(-\Delta_D^\Omega + 1)^{-1} \) is compact in \( L^2(\Omega) \) and the same spectral result holds thus for the operator \( -\Delta_D^\Omega \) we are interested in. \( \square \)
Integrating the last term by parts in $u$ we conclude that for any $\varepsilon > 0$ there is $D := D_g \otimes \{\chi_1\} \subset C^\infty_0(\Omega)$ such that

$$\forall \psi \in D : \|\nabla \psi\|^2 - (\psi, K_u \psi) \leq (\kappa_1^2 + (C_+/C_-^2) \varepsilon^2) \|\psi\|^2,$$

where $K_u := \frac{K}{1 - 2M u + K u^2}$ is the Gauss curvature of the above indicated parallel surface.

This proves $\inf \sigma_{\text{ess}}(-\Delta_D - K_u) \leq \kappa_1^2$. Since $K_u$ vanishes at infinity by assumption, the operator $K_u(-\Delta_D + 1)^{-1}$ is compact in $L^2(\Omega)$ and the same spectral result holds thus for the operator $-\Delta_D$ we are interested in $\square$

**Remark:** Notice that only $K \to 0$ at infinity is needed in order to establish the upper bound.
Surfaces without poles

We needed geodetical polar coordinates to construct mollifiers in our trial functions. This can be circumvented:

**Lemma** [Carron-E.-Krejčiřík, 2004]: Assume $\langle \Sigma 1 \rangle$, then there is a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ of smooth functions with compact supports in $\Sigma$ such that

- $\forall n \in \mathbb{N} : 0 \leq \varphi_n \leq 1$
- $\| \nabla_g \varphi_n \|_g \to 0$ as $n \to \infty$
- $\varphi_n \to 1$ as $n \to \infty$ uniformly on compacts of $\Sigma$
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**Proof:** Under $\langle \Sigma 1 \rangle$ a classical result of [Huber ’57] states that $(\Sigma, g)$ is conformally equivalent to a closed surface with a finite number of points removed. However, the integral $\|\nabla_g \varphi_n\|_g$ is a conformal invariant and it is easy to find a sequence having the required properties on the “pierced” closed surface. □
Theorem [Carron-E.-Krejčiřík, 2004]: Under the stated assumptions, one has $\inf \sigma(-\Delta^\Omega_D) < \kappa_1^2$ whenever $\Sigma$ is not conformally equivalent to the plane.
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**Proof:** Indeed, the Cohn-Vossen inequality yields

$$\mathcal{K} \leq 2\pi (2 - 2h - e),$$

where $h$ is the genus of $\Sigma$ and $e$ is the number of ends. Hence $\mathcal{K} < 0$ whenever $h \geq 1$. $\Box$
Layers over $\Sigma$ with cylindrical ends

**Theorem** [Carron-E.-Krejčiřík, 2004]: Assume $\langle \Omega_0 \rangle$, $\langle \Omega_1 \rangle$, $\langle \Sigma_0 \rangle$ and $\langle \Sigma_1 \rangle$. Let the reference surface $\Sigma$ have $N \geq 1$ **cylindrically symmetric ends**, each with a positive total Gauss curvature. Let $\Omega' \subset \mathbb{R}^3$ be an unbounded, without boundary, obtained by a compact deformation of $\Omega$. Then

- $\inf \sigma_{\text{ess}}(-\Delta_{\Omega'}) = \kappa_1^2$
- there is at least $N$ ev’s in $\left(0, \kappa_1^2\right)$, counting multiplicity
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**Sketch of the proof:** Deriving the sufficient condition for cylindrical surfaces with $\mathcal{K} > 0$; we constructed sequences of trial functions “localised at infinity” we may use them for our $\Omega$. Moreover, trial functions localized at different ends are orthogonal in $L^2(\Omega)$. Finally, these estimates as well as $\sigma_{\text{ess}}$ are stable under compact deformations of $\Omega$. \qed
Layers with ends: examples

Layer over $\Sigma$ with multiple ends:
Layers with ends: examples

- **Layer over $\Sigma$ with multiple ends:**

- **Conical layer:**
Weak coupling: preliminaries

Consider mildly curved quantum layers generated by a family of surfaces \( \Sigma_\varepsilon := p(\mathbb{R}^2) \) given by a Monge patch

\[
p : \mathbb{R}^2 \to \mathbb{R}^3, \quad p(x^1, x^2; \varepsilon) := (x^1, x^2, \varepsilon f(x^1, x^2))
\]

with \( f \in C^4 \) and ask what happens in the asymptotics \( \varepsilon \to 0 \).
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**Regularity and decay assumptions:**

\[
\langle d1, 4 \rangle f,\mu, f,\mu\nu\rho\sigma \in L^\infty(\mathbb{R}^2)
\]

\[
\langle d2, 3 \rangle f,\mu\nu, f,\mu\nu\rho \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty
\]

They ensure, in particular, that \( \inf \sigma_{\text{ess}}(-\Delta_D^{\Omega_\varepsilon}) = \kappa_1^2 \)
Weak coupling: preliminaries

Consider mildly curved quantum layers generated by a family of surfaces \( \Sigma_\varepsilon := p(\mathbb{R}^2) \) given by a Monge patch

\[
p : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad p(x^1, x^2; \varepsilon) := (x^1, x^2, \varepsilon f(x^1, x^2))
\]

with \( f \in C^4 \) and ask what happens in the asymptotics \( \varepsilon \rightarrow 0 \)

**Regularity and decay assumptions:**

\[
\langle d1, 4 \rangle f,_{\mu}, f,_{\mu\nu\rho\sigma} \in L^\infty(\mathbb{R}^2)
\]

\[
\langle d2, 3 \rangle f,_{\mu\nu}, f,_{\mu\nu\rho} \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty
\]

They ensure, in particular, that \( \inf \sigma_{\text{ess}}(-\Delta_{D}^{\Omega_\varepsilon}) = \kappa_1^2 \)

**Integral decay assumptions:**

\[
\langle r1, 2 \rangle f,_{\mu\nu}, f,_{\mu\nu\rho} \in L^2(\mathbb{R}^2, (1 + |x|^\delta) \, dx)
\]

\[
\langle r3 \rangle f,_{\mu\nu\rho\sigma} \in L^1(\mathbb{R}^2, (1 + |x|^\delta) \, dx) \quad \text{for some} \quad \delta > 0
\]
Weak coupling: asymptotic expansion

**Theorem** [E.-Krejčiřík, 2001]: Let $\Omega_\varepsilon$ be layers generated by $\Sigma_\varepsilon$ with $f \in C^4(\mathbb{R}^2)$ satisfying $\langle d1-4 \rangle$ and $\langle r1-3 \rangle$. If $\Sigma_1$ is not planar, then for all $\varepsilon$ small enough $-\Delta_{D}^{\Omega_\varepsilon}$ has exactly one isolated eigenvalue $E(\varepsilon)$ below the essential spectrum, and

$$E(\varepsilon) = \kappa_1^2 - e^{2w(\varepsilon)^{-1}},$$

where $w(\varepsilon)$ has the following asymptotic expansion

$$w(\varepsilon) = -\varepsilon^2 \sum_{j=2}^{\infty} (\chi_j, u \chi_j) (\kappa_j^2 - \kappa_1^2)^2 \int_{\mathbb{R}^2} \frac{|\mathring{m}_0(\omega)|^2}{|\omega|^2 + \kappa_j^2 - \kappa_1^2} \, d\omega + \mathcal{O}(\varepsilon^{2+\gamma})$$

with $\gamma := \min\{1, \delta/2\}$. Here $m_0$ is the lowest-order term in the expansion of the mean curvature of $\Sigma_\varepsilon$ w.r.t. $\varepsilon$. 
Remarks

The sum in the asymptotic expansion runs in fact over even \( n \) only because one integrates over \((-a, a)\) on which \( u \mapsto \chi_1(u) u \chi_j(u) \) is odd for odd \( j \).
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**The leading-term coefficient** \( w_1 \) in the expansion \( w(\varepsilon) =: \varepsilon^2 w_1 + \mathcal{O}(\varepsilon^{2+\gamma}) \) does not have a very transparent structure. For thin layers it can be rewritten as

\[
    w_1 = -\frac{1}{2\pi} \|m_0\|^2 \frac{\pi^2 - 6}{24\pi^3} \|\nabla m_0\|^2 \delta^2 + \mathcal{O}(d^4),
\]

which is instructive because the first term comes from the surface attractive potential \( K - M^2 \) which dominates the picture in this case.
Birman-Schwinger analysis

Let $M \subset \mathbb{R}^m$, $m \geq 1$, be open connected precompact; put

$$H_\lambda = -\Delta_D + \lambda V$$

with $\lambda > 0$ on $\mathcal{H} := L^2(\mathbb{R}^2) \otimes L^2(M)$

where $-\Delta_D$ is the closure of $-\Delta \otimes I_m + I_2 \otimes -\Delta^M_D$
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Assumptions:

$$\langle a0 \rangle \quad \inf \sigma_{\text{ess}}(H_\lambda) \geq \kappa^2_1$$

$$\langle a1 \rangle \quad \exists \ a, b \geq 0 \quad \forall \psi \in W^{1,2}_0(\Omega_0) : \quad \|V \psi\| \leq a\|\psi\| + b \|H_0^{1/2} \psi\|$$

$$\langle a2 \rangle \quad |V|_{11} \in L^{1+\delta}(\mathbb{R}^2)$$

$$\langle a3 \rangle \quad |V|_{11} \in L^1(\mathbb{R}^2, (1 + |x|^\delta) \, dx)$$

where $V_{jj'} := \int_M \bar{\chi}_j(y) \, V(\cdot, y) \, \chi_{j'}(y) \, dy$ w.r.t. transverse basis of ef’s $\chi_j$, $j = 1, 2, \ldots$ with ev’s $\kappa^2_1 < \kappa^2_2 \leq \ldots \leq \kappa^2_{j} < \ldots$
Birman-Schwinger analysis

The free resolvent operator can be rewritten as

\[ R_0(\alpha) = \sum_{j=1}^{\infty} \chi_j \left( -\Delta + k_j(\alpha)^2 \right)^{-1} \bar{\chi}_j, \quad k_j(\alpha) := \sqrt{\kappa_j^2 - \alpha^2} \]

We are interested in ev’s below \( \kappa_1^2 \), i.e. \( \alpha \in [0, \kappa_1) \), when

\[ R_0(x, y, x', y'; \alpha) = \frac{1}{2\pi} \sum_{j=1}^{\infty} \chi_j(y) K_0 \left( k_j(\alpha)|x - x'| \right) \bar{\chi}_j(y') \]
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Define \( K(\alpha) := |V|^{1/2} R_0(\alpha) V^{1/2} \), where \( V^{1/2} := |V|^{1/2}\text{sgn} V \).

By *Birman-Schwinger principle* \( \alpha(\lambda)^2 \equiv E(\lambda) \) is an ev of \( H_\lambda \) iff \( \lambda K(\alpha) \) has eigenvalue \(-1\), in other words

\[ \alpha^2 \in \sigma_{\text{disc}}(H_\lambda) \iff -1 \in \sigma_{\text{disc}}(\lambda K(\alpha)) \]
BS analysis: decomposition

One has to split the logarithmic singularity responsible for the weakly coupled ev. Put $K(\alpha) = L_\alpha + M_\alpha$, where

$$L_\alpha(x, y, x', y') := -\frac{1}{2\pi} |V(x, y)|^{1/2} \chi_1(y) \ln k_1(\alpha) \chi_1(y') V(x', y')^{1/2}$$

contains the singularity and $M_\alpha$ splits into two parts again, $M_\alpha = A_\alpha + B_\alpha$ with $B_\alpha$ being the projection of resolvent onto higher transverse modes, $j \geq 2$. 
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On the other hand, the operator $A_\alpha$ has the kernel

$$\frac{1}{2\pi} |V(x, y)|^{1/2} \chi_1(y) \left(K_0(k_1(\alpha)|x - x'|) + \ln k_1(\alpha)\right) \chi_1(y') V(x', y')^{1/2}$$

Note that $M_\alpha$ is well defined for $\alpha = \kappa_1$
Using asymptotic behaviour of $K_0$ we deduce

**Lemma** [E.-Krejčiřík, 2001]: Assume $\langle a1-3 \rangle$, then there are positive $C_2, C_3$ and $C_4$ such that

- $\forall \alpha \in [0, \kappa_1] : \| M_\alpha \| < C_2$
- $\| M_\alpha - M_{\kappa_1} \| \leq C_3 \lambda^\gamma$ with $\gamma := \min\{1, \delta/2\}$,
- $\left\| \frac{dM_\alpha(w)}{dw} \right\| < C_4 |w|^{-1}$ for $\lambda$ small enough, $w := (\ln k_1(\alpha))^{-1}$
BS analysis: eliminating regular part

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3. $\left\| \frac{dM_\alpha(w)}{dw} \right\| < C_4 |w|^{-1}$ for $\lambda$ small enough, $w := (\ln \kappa_1(\alpha))^{-1}$

Next we employ the factorization

$$(I + \lambda K(\alpha))^{-1} = [I + \lambda (I + \lambda M_\alpha)^{-1} L_\alpha]^{-1} (I + \lambda M_\alpha)^{-1}$$

By the lemma we have $\| \lambda M_\alpha \| < 1$ for small $\lambda$, the second factor is invertible and the singularities are determined by the first one
BS analysis: eliminating regular part

Observe that \( \lambda (I + \lambda M_\alpha)^{-1} L_\alpha \) is rank-one operator of the form \((\psi, \cdot)\varphi\), where

\[
\psi(x, y) := -\frac{\lambda}{2\pi} \ln k_1(\alpha) V(x, y)^{1/2} \chi_1(y),
\]

\[
\varphi(x, y) := [(I + \lambda M_\alpha)^{-1} |V|^{1/2} \chi_1](x, y),
\]

so it has just one eigenvalue \((\psi, \varphi)\)
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$$

so it has just one eigenvalue $(\psi, \varphi)$

If the latter should equal $-1$ we get the implicit equation

$$
w = F(\lambda, w), \quad F(\lambda, w) := \frac{\lambda}{2\pi} \left( V^{1/2} \chi_1, (I + \lambda M_\alpha(w))^{-1} |V|^{1/2} \chi_1 \right)
$$

with variable $w$ related to the energy via $\alpha^2 = \kappa_1^2 - e^{2w^{-1}}$
BS analysis: main result

**Theorem** [E.-Krejčířík, 2001]: Assume $\langle a0-3 \rangle$ and $V \neq 0$, then $H_\lambda$ has for small enough $\lambda > 0$ exactly one ev $E(\lambda)$ iff

$$\int_{\mathbb{R}^2} V_{11}(x) \, dx \leq 0$$

and in this case we can have $E(\lambda) = \kappa_1^2 - e^{2w(\lambda)^{-1}}$, where

$$w(\lambda) = \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} V_{11}(x) \, dx$$

$$+ \left( \frac{\lambda}{2\pi} \right)^2 \left\{ \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{11}(x) \left( \gamma_E + \ln \frac{|x-x'|}{2} \right) V_{11}(x') \, dx \, dx' \right. \right.$$  

$$- \sum_{j=2}^{\infty} \int_{\mathbb{R}^2 \times \mathbb{R}^2} V_{1j}(x) K_0(k_j(\kappa_1)|x-x'|) V_{j1}(x') \, dx \, dx' \right\} + O(\lambda^{2+\gamma})$$

with $\gamma := \min\{1, \delta/2\}$
Application to mildly curved layers

For the family of surfaces under consideration we have

\[ g_{\mu\nu}(\varepsilon) = \delta_{\mu\nu} + \varepsilon^2 \eta_{\mu\nu}, \quad (\eta_{\mu\nu}) := \begin{pmatrix} f_{,1}^2 & f_{,1} f_{,2} \\ f_{,1} f_{,2} & f_{,2}^2 \end{pmatrix} \]

\[ g(\varepsilon) := \det(g_{\mu\nu}) = 1 + \varepsilon^2 \text{tr}(\eta_{\mu\nu}) = 1 + \varepsilon^2 (f_{,1}^2 + f_{,2}^2) \]

\[ h_{\mu\nu}(\varepsilon) = \varepsilon g(\varepsilon)^{-\frac{1}{2}} \theta_{\mu\nu}, \quad (\theta_{\mu\nu}) := \begin{pmatrix} f_{,11} & f_{,12} \\ f_{,21} & f_{,22} \end{pmatrix} \]
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This gives, in particular, the curvatures

\[ K(\varepsilon) = \delta_{\mu\nu} \varepsilon^2 g(\varepsilon)^{-2} k_0, \quad k_0 := \det(\theta_{\mu\nu}) = f_{,11} f_{,22} - f_{,12}^2 \]

\[ M(\varepsilon) = \varepsilon g(\varepsilon)^{-\frac{3}{2}} (m_0 + \varepsilon^2 m_1), \quad m_0 := \frac{1}{2} \text{tr}(\theta_{\mu\nu}) = \frac{1}{2} (f_{,11} + f_{,22}) \]

\[ m_1 := \frac{1}{2} \text{tr}(\theta_{\mu\rho} \tilde{\eta}^{\rho\nu}) = \frac{1}{2} (f_{,1}^2 f_{,22} + f_{,2}^2 f_{,11} - 2f_{,1} f_{,2} f_{,12}) \]
Application to mildly curved layers

Now we apply the BS result, estimating the Hamiltonian by

\[ H_- \leq H \leq H_+ \quad \text{with} \quad H_\pm := -\Delta - \partial_3^2 + \varepsilon V_\pm, \]

where

\[ V_\pm(x, u) := \frac{1}{\varepsilon} \left( \frac{C_\pm}{C_\mp^2} u_1 + V_2 \right) (x/\sigma_\pm, u) \]

with \( \sigma_\pm^2 := c_\mp^3 C_\mp^2 / (c_\pm^2 C_\pm) \), where \( c_\pm := 1 \pm \varepsilon^2 \| \eta_{\mu\nu} \|. \)
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with \( \sigma^2_\pm := c_\mp^3 C_\pm^2/(c^2_\pm C_\mp) \), where \( c_\pm := 1 \pm \varepsilon^2 \|\eta_{\mu\nu}\| \).

Furthermore, \( V_2 = \frac{K - M^2}{(1 - 2Mu + Ku^2)^2} \) is as before and

\[ v_1 := -\frac{|u^2 \nabla_g K - 2u \nabla_g M|^2}{4(1 - 2Mu + Ku^2)^2} + \frac{u^2 \Delta_g K - 2u \Delta_g M}{2(1 - 2Mu + Ku^2)} \]

Since \( v_1 \) and \( V_2 \) are \( \varepsilon \)-dependent, \( V_\pm \) are well defined even for \( \varepsilon = 0 \). Expansion in \( \varepsilon \) yields the announced result.
Weak coupling: main result again

**Theorem [E.-Krejčiřík, 2001]:** Let $\Omega_\varepsilon$ be layers generated by $\Sigma_\varepsilon$ with $f \in C^4(\mathbb{R}^2)$ satisfying $\langle d1-4 \rangle$ and $\langle r1-3 \rangle$. If $\Sigma_1$ is not planar, then for all $\varepsilon$ small enough $-\Delta_{D}^{\Omega_\varepsilon}$ has exactly one isolated eigenvalue $E(\varepsilon)$ below the essential spectrum, and

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$$w(\varepsilon) = -\varepsilon^2 \sum_{j=2}^{\infty} (\chi_1, u\chi_j) \left( \kappa_j^2 - \kappa_1^2 \right)^2 \int_{\mathbb{R}^2} \frac{|\widehat{m}_0(\omega)|^2}{|\omega|^2 + \kappa_j^2 - \kappa_1^2} d\omega + O(\varepsilon^{2+\gamma})$$

with $\gamma := \min\{1, \delta/2\}$. Here $m_0$ is the lowest-order term in the expansion of the mean curvature of $\Sigma_\varepsilon$ w.r.t. $\varepsilon$.
Open questions

- **Existence for \( \mathcal{K} > 0 \):** recently Lu-Lin announced proof for ends which are graphs of a convex function. 
  
  More generally: when does \( \mathcal{K} > 0 \) imply \( \mathcal{M} = \infty \)?
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- **Layers with non-smooth boundary:** existence proofs, mode matching, examples
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- **Perturbation theory** with respect to various parameters, in particular, the layer thickness
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- **Periodically curved layers:** absolute continuity of the spectrum, existence of gaps

- **More questions:** layers with magnetic fields, regular and singular potential perturbations, etc.
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