Unstable system dynamics: do we understand it fully?

Pavel Exner,
in part together with Martin Fraas, Takashi Ichinose, Sylwia Kondej
Hagen Neidhardt and Valentin Zagrebnov

exner@ujf.cas.cz

Doppler Institute
for Mathematical Physics and Applied Mathematics
Prague
Motivation: quantum kinematics of decays, with and without repeated measurements
Talk overview

- **Motivation:** quantum kinematics of decays, with and without repeated measurements
- **Zeno dynamics:** existence, form of the generator
Talk overview

- **Motivation**: quantum kinematics of decays, with and without repeated measurements
- **Zeno dynamics**: existence, form of the generator
- **Anti-Zeno effect**: what is it and under which conditions it can occur?
Talk overview

- **Motivation:** quantum kinematics of decays, with and without repeated measurements
- **Zeno dynamics:** existence, form of the generator
- **Anti-Zeno effect:** what is it and under which conditions it can occur?
- **Solvable model:** a caricature description of a system of a quantum wire and dots
Talk overview

**Motivation:** quantum kinematics of decays, with and without repeated measurements

**Zeno dynamics:** existence, form of the generator

**Anti-Zeno effect:** what is it and under which conditions it can occur?

**Solvable model:** a caricature description of a system of a quantum wire and dots

**Comparison:** relations between the stable and Zeno dynamics in this and other models
Motivation: quantum kinematics of decays, with and without repeated measurements

Zeno dynamics: existence, form of the generator

Anti-Zeno effect: what is it and under which conditions it can occur?

Solvable model: a caricature description of a system of a quantum wire and dots

Comparison: relations between the stable and Zeno dynamics in this and other models

Regularity of “undisturbed” decay law: example of the Winter model
Quantum kinematics of decays

Three objects are needed:

- the state space $\mathcal{H}$ of an *isolated system*
- projection $P$ to subspace $P\mathcal{H} \subset \mathcal{H}$ of *unstable system*
- *time evolution* $e^{-iHt}$ on $\mathcal{H}$, not reduced by $P$ for $t > 0$
Quantum kinematics of decays

Three objects are needed:

- the state space $\mathcal{H}$ of an *isolated system*
- projection $P$ to subspace $PH \subset \mathcal{H}$ of *unstable system*
- *time evolution* $e^{-iHt}$ on $\mathcal{H}$, not reduced by $P$ for $t > 0$

Suppose that evolution starts at $t = 0$ from a state $\psi \in PH$ and we perform a *non-decay measurement* at some $t > 0$

The non-decay probabilities define in this situation the *decay law*, i.e. the function $P : \mathbb{R}_+ \rightarrow [0, 1]$ defined by

$$P(t) := \|P e^{-iHt} \psi\|^2;$$

we may also denote it as $P_\psi(t)$ to indicate the initial state
Repeated measurements

Suppose we perform non-decay measurements at times $t/n, 2t/n \ldots, t$, all with the positive outcome, then the resulting non-decay probability is

$$M_n(t) = P_\psi(t/n)P_{\psi_1}(t/n) \cdots P_{\psi_{n-1}}(t/n),$$

where $\psi_{j+1}$ is the normalized projection of $e^{-iHt/n}\psi_j$ on $P\mathcal{H}$ and $\psi_0 := \psi$, in particular, for $\dim P = 1$ we have

$$M_n(t) = (P_\psi(t/n))^n.$$
Repeated measurements

Suppose we perform non-decay measurements at times \( t/n, \ 2t/n \ldots, t \), all with the positive outcome, then the resulting non-decay probability is

\[
M_n(t) = P_\psi(t/n)P_{\psi_1}(t/n) \cdots P_{\psi_{n-1}}(t/n),
\]

where \( \psi_{j+1} \) is the normalized projection of \( e^{-iHt/n}\psi_j \) on \( PH \) and \( \psi_0 := \psi \), in particular, for \( \dim P = 1 \) we have

\[
M_n(t) = (P_\psi(t/n))^n
\]

Consider the limit of permanent measurement, \( n \to \infty \). If \( \dim P = 1 \) and the one-sided derivative \( \dot{P}(0+) \) vanishes, we find \( M(t) := \lim_{n \to \infty} M_n(t) = 1 \) for all \( t > 0 \), or Zeno effect. The same is true if \( \dim P > 1 \) provided the derivative \( \dot{P}_\psi(0+) \) has such a property for any \( \psi \in PH \).
When does Zeno effect occur?

Recall first a simple (and very old) result:

**Theorem [E.-Havlíček, 1973]:** \( \dot{P}_\psi(0+) = 0 \) holds whenever \( \psi \in Q(H) \)
When does Zeno effect occur?

Recall first a simple (and very old) result:

**Theorem [E.-Havlíček, 1973]:** \( \dot{P}_\psi(0+) = 0 \) holds whenever \( \psi \in Q(H) \)

**Remarks:**

- Naturally, \( M(t) = P(t) \) if the undisturbed decay law is exponential, i.e. \( P(t) = e^{-\Gamma t} \)

- However, \( P(t) = e^{-\Gamma t} \) correspond to a state not belonging to \( Q(H) \). And what is worse, decay law exponentiality requires \( \sigma(H) = \mathbb{R} \)!
The effect first recognized in [Beskow-Nilsson’67], at least in the non-decay measurement context.
Zeno effect: a bit of history

- The effect first recognized in [Beskow-Nilsson’67], at least in the non-decay measurement context
- Mathematically first established by Friedmann and Chernoff in the beginning of the 70’s
Zeno effect: a bit of history

- The effect first recognized in [Beskow-Nilsson’67], at least in the non-decay measurement context
- Mathematically first established by Friedmann and Chernoff in the beginning of the 70’s
- Its popularity followed the paper [Misra-Sudarshan’77] where the name *quantum Zeno effect* was coined
Zeno effect: a bit of history

- The effect first recognized in [Beskow-Nilsson’67], at least in the non-decay measurement context
- Mathematically first established by Friedmann and Chernoff in the beginning of the 70’s
- Its popularity followed the paper [Misra-Sudarshan’77] where the name quantum Zeno effect was coined
- New interest in recent years, in particular, because the effect becomes experimentally accessible in its non-ideal form: lifetime enhancement by measurement. Moreover, even practical applications are envisioned
Zeno effect: a bit of history

The effect first recognized in [Beskow-Nilsson’67], at least in the non-decay measurement context.

Mathematically first established by Friedmann and Chernoff in the beginning of the 70’s.

Its popularity followed the paper [Misra-Sudarshan’77] where the name quantum Zeno effect was coined.

New interest in recent years, in particular, because the effect becomes experimentally accessible in its non-ideal form: lifetime enhancement by measurement. Moreover, even practical applications are previsioned.

New mathematical questions, in particular, about Zeno dynamics: what is the time evolution in $\mathcal{PH}$ generated by permanent observation?
Assume that $H$ is \textit{bounded from below} and consider the non-trivial situation, $\dim \mathcal{H} > 1$. We ask: does the limit

$$
(P e^{-iHt/n} P)^n \longrightarrow e^{-iH_P t}
$$

hold as $n \rightarrow \infty$, in which sense, and what is then Zeno dynamics generator, i.e. the operator $H_P$?
Assume that $H$ is bounded from below and consider the non-trivial situation, $\dim \mathcal{H} > 1$. We ask: does the limit

$$(Pe^{-iHt/n}P)^n \longrightarrow e^{-iH_pt}$$

hold as $n \rightarrow \infty$, in which sense, and what is then Zeno dynamics generator, i.e. the operator $H_P$?

Consider quadratic form $u \mapsto \|H^{1/2}Pu\|^2$ with the form domain $D(H^{1/2}P)$ which is closed. By [Chernoff’74] the associated s-a operator, $(H^{1/2}P^*)(H^{1/2}P)$, is a natural candidate for $H_P$ (while, in general, $PHP$ is not!)

Counterexamples in [E.’85] and [Matolcsi-Shvidkoy’03] show, however, that it is necessary to assume that $H$ is densely defined.
Zeno dynamics, continued

**Proposition:** Let $H$ be a self-adjoint operator in a separable $\mathcal{H}$, bounded from below, and let $P$ be a finite-dimensional orthogonal projection on $\mathcal{H}$. If $PH \subset Q(H)$, then for any $\psi \in \mathcal{H}$ and $t \geq 0$ we have

$$\lim_{n \to \infty} (Pe^{-iHt/n}P)^n \psi = e^{-iH_P t} \psi,$$

uniformly on any compact interval of the variable $t$.
**Proposition:** Let \( H \) be a self-adjoint operator in a separable \( \mathcal{H} \), bounded from below, and let \( P \) be a finite-dimensional orthogonal projection on \( \mathcal{H} \). If \( PH \subset Q(H) \), then for any \( \psi \in \mathcal{H} \) and \( t \geq 0 \) we have

\[
\lim_{n \to \infty} (Pe^{-iHt/n}P)^n \psi = e^{-iHPt} \psi,
\]

uniformly on any compact interval of the variable \( t \).

**Proof (following Graf & Guekos):** (i) We need to check

\[
\lim_{t \to 0} t^{-1} \left\| Pe^{-itH}P - Pe^{-itHP} P \right\| = 0,
\]

since it implies \( \left\| (Pe^{-itH/n}P)^n - e^{-itHP} \right\| = n o(t/n) \) as \( n \to \infty \) by means of a natural telescopic estimate.
Zeno dynamics, continued

One may assume $H \geq c I, \ c > 0$. First we first prove that

$$t^{-1} \left[ (f, Pe^{-itH} Pg) - (f, g) - it(\sqrt{HP}f, \sqrt{HP}g) \right] \longrightarrow 0$$

as $t \to 0$ for all $f, g$ from $D(\sqrt{HP}) = PH$. The LHS equals

$$(\sqrt{HP}f, \left[ \frac{e^{-itH}}{tH} - I \right] \sqrt{HP}g)$$

and the square bracket tends to zero strongly.
Zeno dynamics, continued

One may assume $H \geq cI$, $c > 0$. First we first prove that

$$t^{-1} \left[ (f, Pe^{-itH} P g) - (f, g) - it(\sqrt{H} P f, \sqrt{H} P g) \right] \rightarrow 0$$

as $t \rightarrow 0$ for all $f, g$ from $D(\sqrt{H} P) = PH$. The LHS equals

$$\left( \sqrt{H} P f, \left[ \frac{e^{-itH}}{tH} - I \right] \sqrt{H} P g \right)$$

and the square bracket tends to zero strongly. In the same way we find that

$$t^{-1} \left[ (f, Pe^{-itH}^2 P g) - (f, g) - it(\sqrt{H} P f, \sqrt{H} P g) \right] \rightarrow 0$$

holds as $t \rightarrow 0$ for any $f, g \in PH$. Next we note that

$$(\sqrt{H} P f, \sqrt{H} P g) = (\sqrt{H} P f, \sqrt{H} P g),$$

and consequently,

$t^{-1}(Pe^{-itH} P - Pe^{-itH}^2 P) \rightarrow 0$ weakly as $t \rightarrow 0$, however, the two topologies are equivalent if $\dim P < \infty$. □
Without the restriction, situation is more complicated:

**Theorem [E.-Ichinose ’04]:** Under same assumptions, except that \( P \) can be arbitrary, we have for any \( T > 0 \)

\[
\lim_{n \to \infty} \int_{0}^{T} \left\| (Pe^{-iHt/n} P)^n \psi - e^{-iH_P t} \psi \right\|^2 dt = 0
\]
Zeno dynamics, continued

Without the restriction, situation is more complicated:

**Theorem [E.-Ichinose ’04]:** Under same assumptions, except that \( P \) can be arbitrary, we have for any \( T > 0 \)

\[
\lim_{n \to \infty} \int_0^T \| (P e^{-iHt/n} P)^n \psi - e^{-iH_P t} \psi \|^2 \, dt = 0
\]

**Theorem [E.-Neidhardt-Ichinose-Zagrebnov ’06]:** Under same assumptions, except that \( \mathcal{H} \) need not be separable

\[
\lim_{n \to \infty} (P E_H([0, \pi n/t]) e^{-iHt/n} P)^n \psi = e^{-iH_P t} \psi,
\]

uniformly on any compact interval of the variable \( t \), and same for \( (P \phi(tH/n) P)^n \) with \( |\phi(x)| \leq 1, \phi(0) = 1, \phi'(0) = -i \).
Zeno dynamics, continued

Without the restriction, situation is more complicated:

**Theorem [E.-Ichinose ’04]:** Under same assumptions, except that $P$ can be arbitrary, we have for any $T > 0$

$$\lim_{n \to \infty} \int_{0}^{T} \left\| \left( P e^{-iHt/n} P \right)^n \psi - e^{-iH_P t} \psi \right\|^2 dt = 0$$

**Theorem [E.-Neidhardt-Ichinose-Zagrebnov ’06]:** Under same assumptions, except that $\mathcal{H}$ need not be separable

$$\lim_{n \to \infty} \left( PE_H([0, \pi n/t]) e^{-iHt/n} P \right)^n \psi = e^{-iH_P t} \psi,$$

uniformly on any compact interval of the variable $t$, and same for $(P \phi(tH/n) P)^n$ with $|\phi(x)| \leq 1$, $\phi(0) = 1$, $\phi'(0) = -i$

**Corollary:** Strong convergence holds provided $\|H\| < \infty$
Measurements again: what is anti-Zeno?

Let us now return to “Zeno-type” non-decay probability, 

\[ M_n(t) = P_\psi(t/n) P_{\psi_1}(t/n) \cdots P_{\psi_{n-1}}(t/n), \]

where \( \psi_{j+1} \) are as before, in particular, to the formula

\[ M_n(t) = (P_\psi(t/n))^n \]

for \( \dim P = 1. \)
Measurements again: what is anti-Zeno?

Let us now return to “Zeno-type” non-decay probability, 
\[ M_n(t) = P_\psi(t/n)P_{\psi_1}(t/n) \cdots P_{\psi_{n-1}}(t/n), \]
where \( \psi_{j+1} \) are as before, in particular, to the formula

\[ M_n(t) = (P_\psi(t/n))^n \]

for \( \dim P = 1 \). Since \( \lim_{n \to \infty} (f(t/n)^n = \exp\{-\dot{f}(0+)t\} \)
if 
\( f(0) = 1 \) and the one-sided derivative \( \dot{f}(0+) \) exists we see that 
\( M(t) := \lim_{n \to \infty} M_n(t) = 0 \) for \( \forall t > 0 \) if \( \dot{P}(0+) = -\infty \),
and the same is true if \( \dim P > 1 \) provided the derivative 
\( \dot{P}_\psi(0+) \) has such a property for any \( \psi \in \mathcal{PH} \).
Measurements again: what is anti-Zeno?

Let us now return to “Zeno-type” non-decay probability, $M_n(t) = P_\psi(t/n)P_\psi_1(t/n) \cdots P_\psi_{n-1}(t/n)$, where $\psi_{j+1}$ are as before, in particular, to the formula

$$M_n(t) = (P_\psi(t/n))^n$$

for $\dim P = 1$. Since $\lim_{n \to \infty} (f(t/n)^n = \exp\{-\dot{f}(0+)t\}$ if $f(0) = 1$ and the one-sided derivative $\dot{f}(0+)$ exists we see that $M(t) := \lim_{n \to \infty} M_n(t) = 0$ for $\forall t > 0$ if $\dot{P}(0+) = -\infty$, and the same is true if $\dim P > 1$ provided the derivative $\dot{P}_\psi(0+)$ has such a property for any $\psi \in PH$.

It is idealization, of course, but validity of such idealizations is the heart and soul of theoretical physics and has the same fundamental significance as the reproducibility of experimental data [Bratelli-Robinson’79].
We need to estimate the quantity $1 - P(t)$, in other words $(\psi, P\psi) - (\psi, e^{iHt} Pe^{-iHt} \psi)$. We rewrite it as

$$1 - P(t) = 2 \text{Re} (\psi, P(I - e^{-iHt})\psi) - \| P(I - e^{-iHt})\psi \|^2$$
Decay probability estimate

We need to estimate the quantity $1 - P(t)$, in other words $(\psi, P\psi) - (\psi, e^{iHt} Pe^{-iHt} \psi)$. We rewrite it as

$$1 - P(t) = 2 \text{Re} (\psi, P(I - e^{-iHt})\psi) - \| P(I - e^{-iHt})\psi \|^2$$

In terms of the spectral measure $E_H$ of $H$ the r.h.s. equals

$$4 \int_{-\infty}^{\infty} \sin^2 \frac{\lambda t}{2} d\| E^H_\lambda \psi \|^2 - 4 \left\| \int_{-\infty}^{\infty} \frac{e^{-i\lambda t/2}}{2} \sin \frac{\lambda t}{2} dPE^H_\lambda \psi \right\|^2$$

By Schwarz it is non-negative; our aim is to find tighter upper and lower bounds. In particular, for $\dim P = 1$ we denote $d\omega(\lambda) := d(\psi, E^H_\lambda \psi)$ obtaining

$$4 \int_{-\infty}^{\infty} \sin^2 \frac{\lambda t}{2} d\omega(\lambda) - 4 \left| \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin \frac{\lambda t}{2} d\omega(\lambda) \right|^2$$
The one-dimensional case

Let first $\dim P = 1$. One can employ the spectral-measure normalization, $\int_{-\infty}^{\infty} d\omega(\lambda) = 1$, to rewrite the decay probability in the following way

$$2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \sin^2 \frac{\lambda t}{2} + \sin^2 \frac{\mu t}{2} \right) d\omega(\lambda) d\omega(\mu)$$

$$-4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos \left( \frac{\lambda - \mu}{2} t \right) \sin \frac{\lambda t}{2} \sin \frac{\mu t}{2} d\omega(\lambda) d\omega(\mu),$$

or equivalently

$$1 - P(t) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2 \frac{(\lambda - \mu)t}{2} d\omega(\lambda) d\omega(\mu)$$

We can thus try to estimate the integrated function
An estimate from above

Take $\alpha \in (0, 2]$. Using $|x|^\alpha \geq |\sin x|^\alpha \geq \sin^2 x$ together with $|\lambda - \mu|^\alpha \leq 2^\alpha (|\lambda|^\alpha + |\mu|^\alpha)$ we infer from the above formula

$$\frac{1 - P(t)}{t^\alpha} \leq 2^{1-\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\lambda - \mu|^\alpha d\omega(\lambda)d\omega(\mu)$$

$$\leq 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|\lambda|^\alpha + |\mu|^\alpha) d\omega(\lambda)d\omega(\mu) \leq 4 \langle |H|^\alpha \rangle_\psi$$
An estimate from above

Take $\alpha \in (0, 2]$. Using $|x|^\alpha \geq |\sin x|^\alpha \geq \sin^2 x$ together with $|\lambda - \mu|^\alpha \leq 2^\alpha (|\lambda|^\alpha + |\mu|^\alpha)$ we infer from the above formula

$$\frac{1 - P(t)}{t^\alpha} \leq 2^{1-\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\lambda - \mu|^\alpha d\omega(\lambda) d\omega(\mu)$$

$$\leq 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|\lambda|^\alpha + |\mu|^\alpha) d\omega(\lambda) d\omega(\mu) \leq 4 \langle |H|^\alpha \rangle \psi$$

Hence $1 - P(t) = \mathcal{O}(t^\alpha)$ if $\psi \in D(|H|^\alpha/2)$. If this is true for some $\alpha > 1$ we have Zeno effect – which is a slightly weaker sufficient condition than the earlier stated one.
An estimate from above

Take $\alpha \in (0, 2]$. Using $|x|^\alpha \geq |\sin x|^\alpha \geq \sin^2 x$ together with $|\lambda - \mu|^\alpha \leq 2^\alpha (|\lambda|^\alpha + |\mu|^\alpha)$ we infer from the above formula

$$\frac{1 - P(t)}{t^\alpha} \leq 2^{1-\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\lambda - \mu|^\alpha d\omega(\lambda) d\omega(\mu)$$

$$\leq 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|\lambda|^\alpha + |\mu|^\alpha) d\omega(\lambda) d\omega(\mu) \leq 4 \langle |H|^\alpha \rangle \psi$$

Hence $1 - P(t) = O(t^\alpha)$ if $\psi \in D(|H|^\alpha/2)$. If this is true for some $\alpha > 1$ we have Zeno effect – which is a slightly weaker sufficient condition than the earlier stated one.

By negation, $\psi \notin D(|H|^{1/2})$ is a necessary condition for the anti-Zeno effect. Notice that in case $\psi \in H_{ac}(H)$ the same follows from Lipschitz regularity, since $P(t) = |\hat{\omega}(t)|^2$ and $\hat{\omega}$ is bd and uniformly $\alpha$-Lipschitz iff $\int_{\mathbb{R}} \omega(\lambda)(1 + |\lambda|^\alpha) d\lambda < \infty$. 

An estimate from below

To find a *sufficient condition* note that for $\lambda, \mu \in [-1/t, 1/t]$ there is a positive $C$ independent of $t$ such that

$$\left| \sin \frac{(\lambda - \mu)t}{2} \right| \geq C|\lambda - \mu|t;$$

one can make the constant explicit but it is not necessary.
An estimate from below

To find a sufficient condition note that for $\lambda, \mu \in [-1/t, 1/t]$ there is a positive $C'$ independent of $t$ such that

$$\left| \sin \left( \frac{\lambda - \mu}{2} t \right) \right| \geq C' |\lambda - \mu| t;$$

one can make the constant explicit but it is not necessary. Consequently, we have the estimate

$$1 - P(t) \geq 2C^2 t^2 \int_{-1/t}^{1/t} d\omega(\lambda) \int_{-1/t}^{1/t} d\omega(\mu)(\lambda - \mu)^2$$

which in turn implies

$$\frac{1 - P(t)}{t} \geq 4C^2 t \left\{ \int_{-1/t}^{1/t} \lambda^2 d\omega(\lambda) \int_{-1/t}^{1/t} d\omega(\lambda) - \left( \int_{-1/t}^{1/t} \lambda d\omega(\lambda) \right)^2 \right\}$$
Sufficient conditions

The AZ effect occurs if the r.h.s. diverges as $t \to 0$, e.g., if

$$\int_{-N}^{N} \lambda^2 d\omega(\lambda) \int_{-N}^{N} d\omega(\lambda) - \left( \int_{-N}^{N} \lambda d\omega(\lambda) \right)^2 \geq cN^\alpha$$

holds for any $N$ and some $c > 0$, $\alpha > 1$
Sufficient conditions

The AZ effect occurs if the r.h.s. diverges as $t \to 0$, e.g., if

$$\int_{-N}^{N} \lambda^2 \, d\omega(\lambda) \int_{-N}^{N} d\omega(\lambda) - \left( \int_{-N}^{N} \lambda \, d\omega(\lambda) \right)^2 \geq cN^\alpha$$

holds for any $N$ and some $c > 0$, $\alpha > 1$

We can also write it in a more compact form: introduce $H_N^\beta := H^\beta E_H(\Delta_N)$ with the spectral cut-off to the interval $\Delta_N := (-N, N)$, in particular, denote $I_N := E_H(-N, N)$. The sufficient condition then reads

$$\left( \langle H_N^2 \rangle_\psi \langle I_N \rangle_\psi - \langle H_N \rangle_\psi^2 \right)^{-1} = o(N) \quad \text{as} \quad N \to \infty$$
More on the one-dimensional case

Remark: Notice that the condition does not require the Hamiltonian $H$ to be unbounded, in contrast to exponential exponential decay; it is enough that the spectral distribution has a slow decay in one direction only.
More on the one-dimensional case

Remark: Notice that the condition does not require the Hamiltonian $H$ to be unbounded, in contrast to exponential decay; it is enough that the spectral distribution has a slow decay in one direction only.

Example: Consider $H$ bd from below and $\psi$ from $\mathcal{H}_{ac}(H)$ s.t. $\omega(\lambda) \approx c\lambda^{-\beta}$ as $\lambda \to +\infty$ for some $c > 0$ and $\beta \in (1, 2)$. While $\int_{-N}^{N} \omega(\lambda) \, d\lambda \to 1$, the other two integrals diverge giving

$$cN^{2-\beta} - c^2 N^{4-2\beta}$$

as the asymptotic behavior of the l.h.s., where the first term is dominating; it gives $\dot{P}(0+) = -\infty$ so AZ effect occurs.
More on the one-dimensional case

Remark: Notice that the condition does not require the Hamiltonian $H$ to be unbounded, in contrast to exponential exponential decay; it is enough that the spectral distribution has a slow decay in one direction only.

Example: Consider $H$ bd from below and $\psi$ from $\mathcal{H}_{ac}(H)$ s.t. $\omega(\lambda) \approx c\lambda^{-\beta}$ as $\lambda \to +\infty$ for some $c > 0$ and $\beta \in (1, 2)$. While $\int_{-N}^{N} \omega(\lambda) \, d\lambda \to 1$, the other two integrals diverge giving

$$cN^{2-\beta} - c^2N^{4-2\beta}$$

as the asymptotic behavior of the l.h.s., where the first term is dominating; it gives $\dot{P}(0+) = -\infty$ so AZ effect occurs.

Remarks: For $\beta > 2$ we have Zeno effect, so the Z-AZ gap is rather narrow! Also, $\beta = 2$ with a cut-off may give rapid oscillations around $t = 0$ obscuring existence of Zeno limit.
Multiple degrees of freedom

Let $\dim P > 1$ and denote by $\{\chi_j\}$ an orthonormal basis in $P \mathcal{H}$. The second term in the decay-probability formula is

$$-4 \sum_m \left| \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin \frac{\lambda t}{2} d(\chi_m, E_H^\lambda \Psi) \right|^2$$
Multiple degrees of freedom

Let \( \dim P > 1 \) and denote by \( \{\chi_j\} \) an orthonormal basis in \( P\mathcal{H} \). The second term in the decay-probability formula is

\[
-4 \sum_m \left| \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin \frac{\lambda t}{2} d(\chi_m, E_{\lambda}^H \psi) \right|^2
\]

We also expand \( \psi = \sum_j c_j \chi_j \) with \( \sum_j |c_j|^2 = 1 \) and denote \( d\omega_{jk}(\lambda) := d(\chi_j, E_{\lambda}^H \chi_k) \), which is real-valued and symmetric w.r.t. index interchange. Using \( d\|E_{\lambda}^H \psi\|^2 = \sum_{jk} \bar{c}_j c_k d\omega_{jk}(\lambda) \) we can cast the decay-probability into the form

\[
1 - P(t) = 4 \sum_{jk} \bar{c}_j c_k \left\{ \int_{-\infty}^{\infty} \sin^2 \frac{\lambda t}{2} d\omega_{jk}(\lambda) \right. \\
- \left. \sum_m \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin \frac{\lambda t}{2} d\omega_{jm}(\lambda) \int_{-\infty}^{\infty} e^{i\mu t/2} \sin \frac{\mu t}{2} d\omega_{km}(\mu) \right\}
\]
Multiple degrees of freedom, contd

If $\dim P = \infty$ one has to check convergence of the series and correctness of interchanging of the summation and integration; it is done by means of Parseval relation.
Multiple degrees of freedom, contd

If \( \dim P = \infty \) one has to check convergence of the series and correctness of interchanging of the summation and integration; it is done by means of Parseval relation

Next we employ normalization, \( \int_{-\infty}^{\infty} d\omega_{jk}(\lambda) = \delta_{jk} \), to derive

\[
1 - P(t) = 2 \sum_{jkm} \bar{c}_j c_k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2 \frac{(\lambda - \mu)t}{2} d\omega_{jm}(\lambda) d\omega_{km}(\mu)
\]

which can be also written concisely as

\[
1 - P(t) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2 \frac{(\lambda - \mu)t}{2} (\psi, dE^H_{\lambda} PdE^H_{\mu} \psi)
\]
General sufficient condition

Since \(|\sin \frac{(\lambda - \mu)t}{2}| \geq C|\lambda - \mu|t| \) holds for \(|\mu t|, |\lambda t| < 1\) we get

\[ 1 - P(t) \geq 2C^2 t^2 \int_{-1/t}^{1/t} \int_{-1/t}^{1/t} (\lambda - \mu)^2 (\psi, dE^H_{\lambda} P dE^H_{\mu} \psi) \]

\[ = 4C^2 t^2 \int_{-1/t}^{1/t} \int_{-1/t}^{1/t} (\lambda^2 - \lambda \mu) (\psi, dE^H_{\lambda} P dE^H_{\mu} \psi) \]

\[ = 4C^2 t^2 \left\{ (\psi, H^2_{1/t} P I_{1/t} \psi) - \| PH_{1/t} \psi \|^2 \right\} \]
General sufficient condition

Since $\left| \sin \left( \frac{\lambda - \mu}{2} t \right) \right| \geq C |\lambda - \mu| t$ holds for $|\mu t|, |\lambda t| < 1$ we get

$$1 - P(t) \geq 2C^2 t^2 \int_{-1/t}^{1/t} \int_{-1/t}^{1/t} (\lambda - \mu)^2 (\psi, dE^H_\lambda P dE^H_\mu \psi)$$

$$= 4C^2 t^2 \int_{-1/t}^{1/t} \int_{-1/t}^{1/t} (\lambda^2 - \lambda \mu) (\psi, dE^H_\lambda P dE^H_\mu \psi)$$

$$= 4C^2 t^2 \left\{ (\psi, H^2_{1/t} P I_{1/t} \psi) - \| P H_{1/t} \psi \|^2 \right\}$$

Let us summarize the results:

**Theorem [E.’05]:** In the above notation, suppose that

$$(\langle H^2_N P I_N \rangle \psi - \| P H_N \psi \|^2)^{-1} = o(N)$$

holds as $N \to \infty$ uniformly w.r.t. $\psi \in P \mathcal{H}$, then the permanent observation causes anti-Zeno effect
An interlude: a caricature model

An idealized description of a *quantum wire* and a family of *quantum dots*. Formally Hamiltonian acts in $L^2(\mathbb{R}^2)$ as

$$H_{\alpha,\beta} = -\Delta - \alpha \delta(x - \Sigma) + \sum_{i=1}^{n} \tilde{\beta}_i \delta(x - y^{(i)}), \quad \alpha > 0,$$

where $\Sigma := \{(x_1,0); \ x_1 \in \mathbb{R}^2\}$ and $\Pi := \{y^{(i)}\}_{i=1}^{n} \subset \mathbb{R}^2 \setminus \Sigma$.
An interlude: a caricature model

An idealized description of a quantum wire and a family of quantum dots. Formally Hamiltonian acts in $L^2(\mathbb{R}^2)$ as

$$H_{\alpha,\beta} = -\Delta - \alpha \delta(x - \Sigma) + \sum_{i=1}^{n} \tilde{\beta}_i \delta(x - y^{(i)}) , \quad \alpha > 0 ,$$

where $\Sigma := \{(x_1,0) ; x_1 \in \mathbb{R}^2\}$ and $\Pi := \{y^{(i)}\}_{i=1}^{n} \subset \mathbb{R}^2 \setminus \Sigma$

Singular interactions defined conventionally through b.c.: we have $\partial_{x_2} \psi(x_1,0+) - \partial_{x_2} \psi(x_1,0-) = -\alpha \psi(x_1,0)$ for the line; around $y^{(i)}$ the wave functions have to behave as

$$\psi(x) = -\frac{1}{2\pi} \log |x - y^{(i)}| L_0(\psi, y^{(i)}) + L_1(\psi, y^{(i)}) + \mathcal{O}(|x - y^{(i)}|),$$

where the generalized b.v. $L_j(\psi, y^{(i)}), \ j = 0, 1$, satisfy

$$L_1(\psi, y^{(i)}) + 2\pi \beta_i L_0(\psi, y^{(i)}) = 0 , \quad \beta_i \in \mathbb{R}$$
We introduce auxiliary Hilbert spaces, $\mathcal{H}_0 := L^2(\mathbb{R})$ and $\mathcal{H}_1 := \mathbb{C}^n$, and trace maps $\tau_j : W^{2,2}(\mathbb{R}^2) \to \mathcal{H}_j$ defined by $\tau_0 f := f \upharpoonright \Sigma$ and $\tau_1 f := f \upharpoonright \Pi$. 
Resolvent by Krein-type formula

- We introduce auxiliary Hilbert spaces, \( \mathcal{H}_0 := L^2(\mathbb{R}) \) and \( \mathcal{H}_1 := \mathbb{C}^n \), and trace maps \( \tau_j : W^{2,2}(\mathbb{R}^2) \to \mathcal{H}_j \) defined by \( \tau_0 f := f \upharpoonright \Sigma \) and \( \tau_1 f := f \upharpoonright \Pi \),

- canonical embeddings of free resolvent \( R(z) \) to \( \mathcal{H}_i \) by \( R_{i,L}(z) := \tau_i R(z) : L^2 \to \mathcal{H}_i \), \( R_{L,i}(z) := [R_{i,L}(z)]^* \), and \( R_{j,i}(z) := \tau_j R_{L,i}(z) : \mathcal{H}_i \to \mathcal{H}_j \), and
Resolvent by Krein-type formula

- We introduce auxiliary Hilbert spaces, \( \mathcal{H}_0 := L^2(\mathbb{R}) \) and \( \mathcal{H}_1 := \mathbb{C}^n \), and trace maps \( \tau_j : W^{2,2}(\mathbb{R}^2) \to \mathcal{H}_j \) defined by \( \tau_0 f := f \upharpoonright \Sigma \) and \( \tau_1 f := f \upharpoonright \Pi \),

- canonical embeddings of free resolvent \( R(z) \) to \( \mathcal{H}_i \) by
  \[
  R_{i,L}(z) := \tau_i R(z) : L^2 \to \mathcal{H}_i, \quad R_{L,i}(z) := [R_{i,L}(z)]^*,
  \]
  and
  \[
  R_{j,i}(z) := \tau_j R_{L,i}(z) : \mathcal{H}_i \to \mathcal{H}_j,
  \]

- operator-valued matrix \( \Gamma(z) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1 \) by
  \[
  \Gamma_{ij}(z) g := -R_{i,j}(z) g \quad \text{for} \quad i \neq j \quad \text{and} \quad g \in \mathcal{H}_j, \\
  \Gamma_{00}(z) f := [\alpha^{-1} - R_{0,0}(z)] f \quad \text{if} \quad f \in \mathcal{H}_0, \\
  \Gamma_{11}(z) \varphi := \left( s_\beta(z) \delta_{kl} - G_z(y^{(k)}, y^{(l)})(1 - \delta_{kl}) \right) \varphi,
  \]

with \( s_\beta(z) := \beta + s(z) := \beta + \frac{1}{2\pi} (\ln \frac{\sqrt{z}}{2i} - \psi(1)) \).
Resolvent by Krein-type formula

To invert it we define the “reduced determinant”

\[ D(z) := \Gamma_{11}(z) - \Gamma_{10}(z)\Gamma_{00}(z)^{-1}\Gamma_{01}(z) : \mathcal{H}_1 \to \mathcal{H}_1, \]
To invert it we define the “reduced determinant”

\[ D(z) := \Gamma_{11}(z) - \Gamma_{10}(z)\Gamma_{00}(z)^{-1}\Gamma_{01}(z) : \mathcal{H}_1 \to \mathcal{H}_1 , \]

then an easy algebra yields expressions for “blocks” of \([\Gamma(z)]^{-1}\) in the form

\[
[\Gamma(z)]^{-1}_{11} = D(z)^{-1} , \\
[\Gamma(z)]^{-1}_{00} = \Gamma_{00}(z)^{-1} + \Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1} , \\
[\Gamma(z)]^{-1}_{01} = -\Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1} , \\
[\Gamma(z)]^{-1}_{10} = -D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1};
\]

thus to determine singularities of \([\Gamma(z)]^{-1}\) one has to find the null space of \(D(z)\).
We can write \( R_{\alpha,\beta}(z) \equiv (H_{\alpha,\beta} - z)^{-1} \) also as a perturbation of the “line only” Hamiltonian \( \tilde{H}_\alpha \) with the resolvent

\[
R_\alpha(z) = R(z) + R_{L0}(z)\Gamma_{00}^{-1}R_{0L}(z)
\]

We define \( R_{\alpha;L1}(z) : \mathcal{H}_1 \rightarrow L^2 \) by \( R_{\alpha;1L}(z)\psi := R_\alpha(z)\psi \upharpoonright \Pi \) for \( \psi \in L^2 \) and \( R_{\alpha;L1}(z) := R_{\alpha;1L}^*(z) \). Then we have the result:
Resolvent by Krein-type formula

We can write $R_{\alpha,\beta}(z) \equiv (H_{\alpha,\beta} - z)^{-1}$ also as a perturbation of the “line only” Hamiltonian $\tilde{H}_{\alpha}$ with the resolvent

$$R_{\alpha}(z) = R(z) + R_{L0}(z)\Gamma_{00}^{-1}R_{0L}(z)$$

We define $R_{\alpha;L1}(z) : \mathcal{H}_1 \to L^2$ by $R_{\alpha;1L}(z)\psi := R_{\alpha}(z)\psi \upharpoonright_{\Pi}$ for $\psi \in L^2$ and $R_{\alpha;L1}(z) := R_{\alpha;1L}^*(z)$. Then we have the result:

**Theorem [E.-Kondej ’04]:** For $z \in \rho(H_{\alpha,\beta})$ with $\text{Im}\, z > 0$ the resolvent $R_{\alpha,\beta}(z) := (H_{\alpha,\beta} - z)^{-1}$ equals

$$R_{\alpha,\beta}(z) = R(z) + \sum_{i,j=0}^{1} R_{L,i}(z)\Gamma(z)^{-1}_{ij}R_{j,L}(z)$$

$$= R_{\alpha}(z) + R_{\alpha;L1}(z)D(z)^{-1}R_{\alpha;1L}(z)$$
Resonance poles

The decay is due to the tunneling between points and line. It is absent if the interaction is “switched off” (i.e., line “put to an infinite distance”); the corresponding free Hamiltonian is $\tilde{H}_\beta := H_{0,\beta}$. It has $m$ eigenvalues, $1 \leq m \leq n$; we assume that they satisfy the condition

$$-\frac{1}{4} \alpha^2 < \epsilon_1 < \cdots < \epsilon_m < 0 \quad \text{and} \quad m > 1,$$

i.e., the embedded spectrum is simple and non-trivial.
The decay is due to the *tunneling between points and line*. It is absent if the interaction is “switched off” (i.e., line “put to an infinite distance”); the corresponding *free Hamiltonian* is \( \tilde{H}_\beta := H_{0,\beta} \). It has \( m \) eigenvalues, \( 1 \leq m \leq n \); we assume that they satisfy the condition

\[
-\frac{1}{4} \alpha^2 < \epsilon_1 < \cdots < \epsilon_m < 0 \quad \text{and} \quad m > 1,
\]

i.e., the embedded spectrum is simple and non-trivial.

Let us specify the interactions sites by their Cartesian coordinates, \( y^{(i)} = (c_i, a_i) \). We also introduce the notations \( a = (a_1, \ldots, a_n) \) and \( d_{ij} = |y^{(i)} - y^{(j)}| \) for the distances in \( \Pi \).

To find resonances in our model we rely on a BS-type argument; our aim is to find zeros of the function \( D(\cdot) \).
Resonance poles, continued

We seek analytic continuation of $D(\cdot)$ across $(-\frac{1}{4}\alpha^2, 0) \subset \mathbb{R}$ denoting it as $D(\cdot)^{(-1)}$. The first component of $\Gamma_{11}(\cdot)^{(l)}$ is obtained easily. To find the second one let us introduce

$$\mu_{ij}(z, t) := \frac{i\alpha}{2^5\pi} \frac{(\alpha - 2i(z - t)^{1/2}) e^{i(z-t)^{1/2}(|a_i|+|a_j|)}}{t^{1/2}(z-t)^{1/2}} e^{it^{1/2}(c_i-c_j)}.$$ 

Then the matrix elements of $(\Gamma_{10}\Gamma^{-1}_{00}\Gamma_{01})(\cdot)(\cdot)$ are

$$\theta_{ij}^{(-1)}(\lambda) = - \int_0^{\infty} \frac{\mu_{ij}^0(\lambda, t)}{t - \lambda - \alpha^2/4} \, dt - 2g_{\alpha,ij}(\lambda)$$

where

$$g_{\alpha,ij}(z) := \frac{i\alpha}{(z + \alpha^2/4)^{1/2}} e^{-\alpha(|a_i|+|a_j|)/2} e^{i(z+\alpha^2/4)^{1/2}(c_i-c_j)};$$

the values at the segment and in $\mathbb{C}_+$ are expressed similarly
Resonance poles, continued

Then we can express $\det D^{(-1)}(z)$. To study weak-coupling asymptotics it is useful to introduce a reparametrization

$$\tilde{b}(a) \equiv (b_1(a), \ldots, b_n(a)), \quad b_i(a) = e^{-|a_i|\sqrt{-\epsilon_i}}$$

denoting the quantity of interest as $\eta(\tilde{b}, z) = \det D^{(-1)}(a, z)$
Resonance poles, continued

Then we can express \( \det D^{-1}(z) \). To study \textit{weak-coupling asymptotics} it is useful to introduce a \textit{reparametrization}

\[
\tilde{b}(a) \equiv (b_1(a), \ldots, b_n(a)), \quad b_i(a) = e^{-|a_i|\sqrt{-\epsilon_i}}
\]

denoting the quantity of interest as \( \eta(\tilde{b}, z) = \det D^{-1}(a, z) \)

If \( \tilde{b} = 0 \) the zeros are, of course, ev’s of the point-interaction Hamiltonian \( \tilde{H}_\beta \). Using implicit-function theorem we find the following weak-coupling asymptotic expansion,

\[
z_i(b) = \epsilon_i + O(b) + iO(b) \quad \text{where} \quad b := \max_{1 \leq i \leq m} b_i
\]
Resonance poles, continued

Then we can express $\det D^{-1}(z)$. To study weak-coupling asymptotics it is useful to introduce a reparametrization

$$\tilde{b}(a) \equiv (b_1(a), \ldots, b_n(a)), \quad b_i(a) = e^{-|a_i|\sqrt{-\epsilon_i}}$$

denoting the quantity of interest as $\eta(\tilde{b}, z) = \det D^{-1}(a, z)$

If $\tilde{b} = 0$ the zeros are, of course, ev’s of the point-interaction Hamiltonian $\tilde{H}_\beta$. Using implicit-function theorem we find the following weak-coupling asymptotic expansion,

$$z_i(b) = \epsilon_i + O(b) + iO(b) \quad \text{where} \quad b := \max_{1 \leq i \leq m} b_i$$

Remark: This model can exhibit also other long-living resonances due to weakly violated mirror symmetry, however, we are not going to consider them here
Dot states

By assumption there is a nontrivial discrete spectrum of $\tilde{H}_\beta$ embedded in $(-\frac{1}{4} \alpha^2, 0)$. Let us denote the corresponding normalized eigenfunctions $\psi_j, j = 1, \ldots, m$, given by

$$\psi_j(x) = \sum_{i=1}^{m} d^{(j)}_i \phi_i^{(j)}(x), \quad \phi_i^{(j)}(x) := \sqrt{-\frac{\epsilon_j}{\pi}} K_0(\sqrt{-\epsilon_j} |x - y^{(i)}|),$$

where vectors $d^{(j)} \in \mathbb{C}^m$ solve the equation $\Gamma_{11}(\epsilon_j) d^{(j)} = 0$ and the normalization condition, $\|\phi_i^{(j)}\| = 1$, reads

$$|d^{(j)}|^2 + 2\text{Re} \sum_{i=2}^{m} \sum_{k=1}^{i-1} d^{(j)}_i d^{(j)}_k (\phi_i^{(j)}, \phi_k^{(j)}) = 1.$$

In particular, if the distances in $\Pi$ are large (the natural length scale is given by $(-\epsilon_j)^{-1/2}$), the cross terms are small and each $|d^{(j)}|$ is close to one.
Decay of the dot states

Now we specify the unstable system identifying its Hilbert space $\mathcal{PH}$ with the span of $\psi_1, \ldots, \psi_m$. If it is prepared at $t = 0$ in a state $\psi \in \mathcal{PH}$, then the undisturbed decay law is

$$P_\psi(t) = \| Pe^{-iH_{\alpha,\beta}t} \psi \|^2$$
Decay of the dot states

Now we specify the unstable system identifying its Hilbert space $P\mathcal{H}$ with the span of $\psi_1, \ldots, \psi_m$. If it is prepared at $t = 0$ in a state $\psi \in P\mathcal{H}$, then the undisturbed decay law is

$$P_\psi(t) = \left\| Pe^{-iH_{\alpha,\beta}t}\psi \right\|^2$$

Our model is similar to (multidimensional) Friedrichs model, therefore modifying the standard argument [Demuth’76], cf. [E.-Ichinose-Kondej’05], one can check that in the weak-coupling situation the leading term in $P_\psi(t)$ will come from the appropriate semigroup evolution on $P\mathcal{H}$, in particular, for the basis states $\psi_j$ we will have a dominantly exponential decay, $P_{\psi_j}(t) \approx e^{-\Gamma_j t}$ with $\Gamma_j = 2 \text{Im } z_j(b)$
Decay of the dot states

Now we specify the unstable system identifying its Hilbert space $\mathcal{PH}$ with the span of $\psi_1, \ldots, \psi_m$. If it is prepared at $t = 0$ in a state $\psi \in \mathcal{PH}$, then the undisturbed decay law is

$$P_\psi(t) = \| Pe^{-iH_{\alpha,\beta}t} \psi \|^2$$

Our model is similar to (multidimensional) Friedrichs model, therefore modifying the standard argument [Demuth’76], cf. [E.-Ichinose-Kondej’05], one can check that in the weak-coupling situation the leading term in $P_\psi(t)$ will come from the appropriate semigroup evolution on $\mathcal{PH}$, in particular, for the basis states $\psi_j$ we will have a dominantly exponential decay, $P_{\psi_j}(t) \approx e^{-\Gamma_j t}$ with $\Gamma_j = 2 \Im z_j(b)$

Remark: The long-time behaviour of $P_{\psi_j}(t)$ is different from Friedrichs model, but this is not important here.
Comparison: stable and Zeno dynamics

Suppose now that we perform the Zeno measurement at our system. We have \( \dim P < \infty \) and \( P \mathcal{H} \subset Q(H_{\alpha,\beta}) \), so \( H_P = PH_{\alpha,\beta}P \) with the following matrix representation

\[
(\psi_j, H_P \psi_k) = \delta_{jk} \epsilon_j - \alpha \int_{\Sigma} \overline{\psi}_j(x_1, 0) \psi_k(x_1, 0) \, dx_1,
\]

where the first term corresponds, of course, to \( \tilde{H}_\beta \).
Comparison: stable and Zeno dynamics

Suppose now that we perform the Zeno measurement at our system. We have $\dim P < \infty$ and $PH \subset Q(H_{\alpha,\beta})$, so $H_P = PH_{\alpha,\beta}P$ with the following matrix representation

$$(\psi_j, H_P \psi_k) = \delta_{jk} \epsilon_j - \alpha \int \bar{\psi}_j(x_1, 0) \psi_k(x_1, 0) \, dx_1,$$

where the first term corresponds, of course, to $\tilde{H}_\beta$

**Theorem [E.-Ichinose-Kondej’05]:** The two dynamics do not differ significantly for times satisfying

$$t \ll C \, e^{2\sqrt{-\epsilon} |\tilde{a}|},$$

where $C$ is a positive number and $|\tilde{a}| = \min_i |a_i|$, $\epsilon = \max_i \epsilon_i$. 
Comparison: stable and Zeno dynamics

Suppose now that we perform the Zeno measurement at our system. We have \( \dim P < \infty \) and \( PH \subset Q(H_{\alpha,\beta}) \), so \( HP = PH_{\alpha,\beta}P \) with the following matrix representation

\[
(\psi_j, HP\psi_k) = \delta_{jk}\epsilon_j - \alpha \int \sum \bar{\psi}_j(x_1,0)\psi_k(x_1,0) \, dx_1,
\]

where the first term corresponds, of course, to \( \tilde{\mathcal{H}}_\beta \).

**Theorem** [E.-Ichinose-Kondej’05]: The two dynamics do not differ significantly for times satisfying

\[
t \ll C \, e^{2\sqrt{-\epsilon|\tilde{a}|}},
\]

where \( C \) is a positive number and \( |\tilde{a}| = \min_i |a_i|, \epsilon = \max_i \epsilon_i \).

**Proof:** The norm of \( \mathcal{U}_t := (e^{-i\tilde{\mathcal{H}}_\beta t} - e^{-iHP t})P \) is small as long as \( t\| (\tilde{\mathcal{H}}_\beta - HP)P \| \ll 1 \); to see when this is true one has to estimate contribution of the cross-terms. \( \square \)
There are more possibilities

It can happen that the \textit{two dynamics are identical}. Choose, e.g., $H_0 := -\Delta_{\Omega}^{D} \oplus -\Delta_{\Omega^c}^{D}$, where $\Omega^c := \mathbb{R}^d \setminus \bar{\Omega}$, and suppose that “switching in” the decay means to remove the Dirichlet barrier between the two complementary regions.

In this case the Zeno-type permanent observation obviously \textit{restores the stable dynamics}
There are more possibilities

It can happen that the \textit{two dynamics are identical}. Choose, e.g., $H_0 := -\Delta^D_{\Omega} \oplus -\Delta^D_{\Omega_c}$, where $\Omega_c := \mathbb{R}^d \setminus \bar{\Omega}$, and suppose that “switching in” the decay means to remove the Dirichlet barrier between the two complementary regions.

In this case the Zeno-type permanent observation obviously \textit{restores the stable dynamics}

On the other hand, the two dynamics \textit{can be different from the outset}. Replace $H_0$ above by the \textit{Neumann} direct sum $-\Delta^N_{\Omega} \oplus -\Delta^N_{\Omega_c}$, so the Zeno and stable time-evolution generators in $\mathcal{PH}$ are $-\Delta^N_{\Omega}$ and $-\Delta^D_{\Omega}$, respectively.

If $\Omega$ is precompact and $\psi$ is Neumann ground state, $\psi(x) = |\Omega|^{-1/2}$, it is unchanged under the stable dynamics while under Zeno one it can evolve into a function which can be \textit{fractal for almost all times} \cite{Berry96, Thaller00}.
Back to “unperturbed” decay

The last example inspires to ask how the “unperturbed” decay law can look like, say, when $\psi$ is not in the domain of the “stable” Hamiltonian.

*Guess:* an irregular behaviour expected when the decay is due to a (weak) tunneling through a potential barrier.
Back to “unperturbed” decay

The last example inspires to ask how the “unperturbed” decay law can look like, say, when $\psi$ is not in the domain of the “stable” Hamiltonian.

Guess: an irregular behaviour expected when the decay is due to a (weak) tunneling through a potential barrier

For definiteness we consider the so-called Winter model,

$$H_\alpha = -\Delta + \alpha \delta(|x| - R), \quad \alpha > 0, \ R > 0;$$

we restrict our attention to the $s$-wave reducing the task to one-dimensional problem having the Hamiltonian on $L^2(\mathbb{R}_+)$

$$h_\alpha = -\frac{d^2}{dr^2} + \alpha \delta(r - R)$$

with $D(h_\alpha) = \{\phi \in W^{2,2}(\mathbb{R}_+) : \phi(0) = 0, \ \phi'(R+) - \phi'(R-) = \alpha \phi(R)\}$.
Decay in Winter model

Using $\psi(\vec{r}, t) = e^{-iH_\alpha t}\psi(\vec{r}, 0)$ and the reduced wave function, $\psi(\vec{r}, t) = \frac{1}{\sqrt{4\pi}} r^{-1} \phi(r, t)$, we can express the decay law as

$$P(t) = \int_0^R |\phi(r, t)|^2 \, dr$$

with the initial state $\phi(\cdot, 0)$ support contained in $B_R(0)$.
Decay in Winter model

Using \( \psi(\vec{r}, t) = e^{-iH_\alpha t}\psi(\vec{r}, 0) \) and the reduced wave function, \( \psi(\vec{r}, t) = \frac{1}{\sqrt{4\pi}} r^{-1} \phi(r, t) \), we can express the decay law as

\[
P(t) = \int_0^R |\phi(r, t)|^2 \, dr
\]

with the initial state \( \phi(\cdot, 0) \) support contained in \( B_R(0) \)

The model is solvable and the time evolution can be expressed through the integral kernel of the resolvent,

\[
e^{-ih_\alpha t} = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_0^\infty e^{-i\lambda t} \text{Im} \frac{1}{h_\alpha - \lambda - i\epsilon} \, d\lambda;
\]

recall that \( \sigma(h_\alpha) = [0, \infty) \) for \( \alpha > 0 \)
Green’s function

The resolvent kernel is given by Krein’s formula,

\[ \frac{1}{\hbar\alpha - k^2} = \frac{1}{\hbar_0 - k^2} + \lambda(k)(\Phi_k, \cdot)\Phi_k(r), \]

where \( \Phi_k(r) = \frac{1}{k} \sin(kr) e^{ikR} \) holds for \( r < R \), and by a direct calculation one finds \( \lambda(k) = -\alpha \left(1 + \frac{i\alpha}{2k}(1 - e^{2ikR})\right)^{-1} \).
Green’s function

The resolvent kernel is given by Krein’s formula,

\[ \frac{1}{\hbar \alpha - k^2} = \frac{1}{\hbar_0 - k^2} + \lambda(k)(\Phi_k, \cdot)\Phi_k(r), \]

where \( \Phi_k(r) = \frac{1}{k} \sin(kr)e^{ikR} \) holds for \( r < R \), and by a direct calculation one finds \( \lambda(k) = -\alpha \left( 1 + \frac{i\alpha}{2k}(1 - e^{2ikR}) \right)^{-1} \).

This gives \( u(t, r, r') = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_0^{\infty} e^{-ik^2t} p(k, r, r') \frac{2k \sin(kr) \sin(kr')}{\pi(2k^2 + 2\alpha^2 \sin^2 kR + 2k\alpha \sin 2kR)} \) for the integral kernel of the evolution operator \( e^{-ih\alpha t} \), where

\[ p(k, r, r') = \frac{2k \sin(kr) \sin(kr')}{\pi(2k^2 + 2\alpha^2 \sin^2 kR + 2k\alpha \sin 2kR)}. \]
Resonance expansion

Singularities of $p(\cdot, r, r')$ are resonances of the problem and their mirror images, $S = \{k_n, -k_n, \bar{k}_n, -\bar{k}_n : n \in \mathbb{N}\}$, around which the function behaves as

$$p(k, r, r') = \frac{i}{2\pi} \frac{v_n(r)v_n(r')}{k^2 - k_n^2} + \chi(k, r, r'),$$

where $v_n$ solves $h\alpha v_n(r) = k_n^2 v_n(r)$ and $\chi$ is locally analytic.
Resonance expansion

Singularities of \( p(\cdot, r, r') \) are resonances of the problem and their mirror images, \( S = \{ k_n, -k_n, \bar{k}_n, -\bar{k}_n : n \in \mathbb{N} \} \), around which the function behaves as

\[
p(k, r, r') = \frac{i}{2\pi} \frac{v_n(r)v_n(r')}{k^2 - k_n^2} + \chi(k, r, r'),
\]

where \( v_n \) solves \( h_\alpha v_n(r) = k_n^2 v_n(r) \) and \( \chi \) is locally analytic.

For \( r, r' < R \) the function \( p(\cdot, r, r') \) decreases in every direction of the \( k \)-plane; thus it can be expressed as sum over the pole singularities

\[
p(k, r, r') = \sum_{\bar{k} \in S} \frac{1}{k - \bar{k}} \text{Res}_{\bar{k}} p(k, r, r')
\]

and by residue theorem we have \( \sum_{\bar{k} \in S} \text{Res}_{\bar{k}} p(k, r, r') = 0 \)
Resonance expansion, continued

Using symmetry of $S$ and $k_n := -\bar{k}_n$ we get

$$p(k, r, r') = \sum_{n \in \mathbb{Z}} \frac{i}{2\pi} \frac{1}{k^2 - k_n^2} \frac{k}{k_n} v_n(r)v_n(r') ,$$

$$\sum_{n \in \mathbb{Z}} \frac{1}{k_n} v_n(r)v_n(r') = 0 ,$$
Resonance expansion, continued

Using symmetry of $S$ and $k_{-n} := -\bar{k}_n$ we get

$$p(k, r, r') = \sum_{n \in \mathbb{Z}} \frac{i}{2\pi} \frac{1}{k^2 - k_n^2} \frac{k}{k_n} v_n(r)v_n(r') ,$$

$$\sum_{n \in \mathbb{Z}} \frac{1}{k_n} v_n(r)v_n(r') = 0 ,$$

and from here the sought kernel is expressed as

$$u(t, r, r') = \sum_{n \in \mathbb{Z}} M(k_n, t)v_n(r)v_n(r')$$

with $M(k_n, t) = \frac{1}{2} e^{u_n^2} \text{erfc} (u_n) $ and $u_n := -e^{-i\pi/4} k_n \sqrt{t}$
Resonance expansion, continued

This yields decay law in the form

\[ P(t) = \sum_{n, l} C_n \bar{C}_l I_{nl} M(k_n, t) \overline{M(k_l, t)} \]

with \( C_n := \int_0^R \phi(r, 0) v_n(r) \, dr \) and \( I_{nl} := \int_0^R v_n(r) \bar{v}_l(r) \, dr \)
Resonance expansion, continued

This yields decay law in the form

\[ P(t) = \sum_{n,l} C_n \bar{C}_l I_{nl} M(k_n, t) \overline{M(k_l, t)} \]

with \( C_n := \int_0^R \phi(r, 0) v_n(r) \, dr \) and \( I_{nl} := \int_0^R v_n(r) \bar{v}_l(r) \, dr \).

To make use of it we need resonance wave functions which are \( v_n(r) = \sqrt{2} Q_n \sin(k_n r) \) with the coefficient \( Q_n \) equal to

\[
\left( \frac{-2ik_n^2}{2k_n + \alpha^2 R \sin 2k_n R + \alpha \sin 2k_n R + 2k_n \alpha R \cos 2k_n R} \right)^{1/2}
\]

Now we can pass to numerical examples choosing \( \alpha = 500 \) using cut-off with \(|n| \leq 1000\) for the series evaluation.
Example: constant in the ball

We choose first $\phi(r, 0) = R^{-3/2} \sqrt{3}r$ for the initial state.

The decay law plot; in the inset we show logarithmic derivative averaged over lengths of about $T/200$. 
Example: integrable singularity at origin

Choose instead \( \phi(r, 0) = R^{-1/2} \) for the initial state which means to start from Neumann ground state on \((0, R)\).

Plotting the same quantities we see a similar behavior.
The wave function plot

For the second example plot the corresponding $|\phi(r, t)|^2$

for $t = T/8$, $T/16$, and $T/27$ (the revival time for $\alpha = \infty$ is $T/8$). The decay modifies the step-function form
A more detailed analysis of \( \dot{P}(t) = -2\text{Im} \left( \phi'(R, t) \bar{\phi}(R, t) \right) \) (equal to flux through the barrier) shows that

If the coefficients in \( \phi(r, t) \approx \sum_n C_n \exp(-ik_n^2 t)v_n(r) \) decay as \( n^{-p} \) with \( p > 1 \) we have \( \dot{P}(t) \to 0 \), uniformly in time, as \( \alpha \to \infty \).
More on decay law derivatives

A more detailed analysis of \( \dot{P}(t) = -2\text{Im} (\phi'(R, t)\bar{\phi}(R, t)) \) (equal to flux through the barrier) shows that

- If the coefficients in \( \phi(r, t) \approx \sum_n C_n \exp(-i k_n^2 t) v_n(r) \) decay as \( n^{-p} \) with \( p > 1 \) we have \( \dot{P}(t) \to 0 \), uniformly in time, as \( \alpha \to \infty \)

- Slow decay: take \( C_n = (-1)^n + 1 \frac{\sqrt{6}}{Rk_n} \) corresponding to our first example, and limit of \( \dot{P}(t_\alpha) \) as \( \alpha \to \infty \) at the moving value \( t_\alpha := t(1 + 2/\alpha R) \). In this case for irrational multiples of \( T \) we find that \( \dot{P}(t) \to 0 \)
More on decay law derivatives

A more detailed analysis of \( \dot{P}(t) = -2\text{Im} \left( \phi'(R, t) \bar{\phi}(R, t) \right) \)
(equal to flux through the barrier) shows that

- If the coefficients in \( \phi(r, t) \approx \sum_n C_n \exp(-ik_n^2 t) v_n(r) \)
decay as \( n^{-p} \) with \( p > 1 \) we have \( \dot{P}(t) \to 0 \), uniformly in time, as \( \alpha \to \infty \)

- **Slow decay**: take \( C_n = (-1)^{n+1} \frac{\sqrt{6}}{Rk_n} \) corresponding to
our first example, and limit of \( \dot{P}(t_\alpha) \) as \( \alpha \to \infty \) at the
moving value \( t_\alpha := t(1 + 2/\alpha R) \). In this case for
**irrational multiples of \( T \)** we find that \( \dot{P}(t) \to 0 \)

- The same is true for \( t = \frac{p}{q} T \) with \( pq \) odd. In contrast,
**for \( pq \) even** we get **nonzero values**, for instance, at the
period we have \( \lim_{\alpha \to \infty} \dot{P}(T_\alpha) = -\frac{4}{3\sqrt{3}} \approx -0.77 \)
Some open questions

Some questions concerning Zeno dynamics remain open; among them, the *natural conjecture* that the Zeno product formula holds in strong operator topology for any semibounded $H$. 
Some open questions

Some questions concerning Zeno dynamics remain open; among them, the *natural conjecture* that the Zeno product formula holds in strong operator topology for any semibounded $H$.

Also, can the formula be valid for *physically interesting* Hamiltonians unbounded from below such as Dirac operators?
Some open questions

Some questions concerning Zeno dynamics remain open; among them, the *natural conjecture* that the Zeno product formula holds in strong operator topology for any semibounded $H$

Also, can the formula be valid for *physically interesting* Hamiltonians unbounded from below such as Dirac operators?

What rigorous claims can be made about “irregular” decays like the one in the Winter model example?
The talk was based on

[EF06] P.E., M. Fraas: The decay law can have an irregular character, quant-ph/0603067


The talk was based on

[EF06] P.E., M. Fraas: The decay law can have an irregular character, quant-ph/0603067


for more information see http://www.ujf.cas.cz/~exner