Nonlinear Eigenvalue Problems and \( \mathcal{PT} \)-Symmetric Hamiltonians

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*Work in progress with Carl M. Bender*
Outline

1. Linear Eigenvalue Problems
2. Painlevé Transcendental Equations I and II
3. Super Painlevé Equations
4. Conclusion and Future Work
Outline

1. **Linear Eigenvalue Problems**
   - Schrödinger equation
   - Separatrix

2. **Painlevé Transcendental Equations I and II**

3. **Super Painlevé Equations**

4. **Conclusion and Future Work**
Schrödinger equation

- Schrödinger equation for the harmonic oscillator:

$$-\frac{1}{2}\psi''(x) + \frac{1}{2}x^2\psi(x) = E\psi(x).$$
Schrödinger equation

- Schrödinger equation for the harmonic oscillator:

\[-\frac{1}{2} \psi''(x) + \frac{1}{2} x^2 \psi(x) = E \psi(x).\]

- Large $x$-behavior:

\[
\psi(x) \sim D_1 \exp \left[ \int_{s_0}^{x} ds \sqrt{s^2 - 2E} \right] \\
+ D_2 \exp \left[ - \int_{s_0}^{x} ds \sqrt{s^2 - 2E} \right], \quad x \to \pm \infty.
\]
Schrödinger equation

- Schrödinger equation for the harmonic oscillator:
  \[-\frac{1}{2}\psi''(x) + \frac{1}{2}x^2\psi(x) = E\psi(x)\].

- Large $x$-behavior:
  \[
  \psi(x) \sim D_1 \exp \left[ \int_x^x ds \sqrt{s^2 - 2E} \right] + D_2 \exp \left[ -\int_x^x ds \sqrt{s^2 - 2E} \right], \quad x \to \pm \infty.
  \]

- The eigenvalue problem: Choose eigenvalues $E$ such that
  \[
  \psi(x) \to 0, \quad \text{for} \quad x \to \pm \infty.
  \]
Eigenstates are (unstable) separatrix solutions
Outline

1. Linear Eigenvalue Problems

2. Painlevé Transcendental Equations I and II
   - Painlevé I: $y''(x) = 6y^2(x) + x$
   - Painlevé II: $y''(x) = 2y^3(x) + xy(x) + \alpha$
   - Relation with $PT$-symmetric Hamiltonians

3. Super Painlevé Equations

4. Conclusion and Future Work
Asymptotic analysis of PI

Three-term equation ⇒ three possible dominant balances.

1. Movable singularities: \( y''(x) \sim 6y^2(x) \gg x, \)

\[
y(x) = \frac{1}{(x - x_0)^2} \left[ 1 + \sum_{n=1}^\infty a_n (x - x_0)^n \right],
\]

where \( x_0 \) and \( a_6 \) are two arbitrary constants ⇒ general solutions.
Asymptotic analysis of PI

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where \( x_0 \) and \( a_6 \) are two arbitrary constants ⇒ general solutions.

2. Large (negative) \( x \) behavior: \( 6y^2(x) \sim -x \gg y''(x), \)

\[
y(x) \sim \pm \sqrt{-\frac{x}{6}}, \quad x \to -\infty.
\]

⇒ Two asymptotes as \( x \to -\infty. \)
Asymptotic analysis of PI

Three-term equation ⇒ three possible dominant balances.

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\]

⇒ Two asymptotes as \( x \to -\infty. \)

3. The third dominant balance, \( y''(x) \sim x \gg y^2(x), \) is inconsistent.
Hyperasymptotic analysis

Let $Y(x)$ be a small difference between two nearby solutions, $Y(x) \equiv y(x) - y_0(x)$, where $y_0(x)$ and $y(x)$ are solutions of PI.
Hyperasymptotic analysis

- Let $Y(x)$ be a small difference between two nearby solutions, $Y(x) \equiv y(x) - y_0(x)$, where $y_0(x)$ and $y(x)$ are solutions of PI.
- To the leading order, $Y(x)$ satisfies a linear differential equation,

$$Y''(x) \sim 12y_0(x)Y(x), \quad x \to -\infty.$$
Hyperasymptotic analysis

- Let $Y(x)$ be a small difference between two nearby solutions, $Y(x) \equiv y(x) - y_0(x)$, where $y_0(x)$ and $y(x)$ are solutions of PI.
- To the leading order, $Y(x)$ satisfies a linear differential equation,
  \[ Y''(x) \sim 12y_0(x)Y(x), \quad x \to -\infty. \]
- Oscillating around the lower asymptote, $y_0(x) \sim -\sqrt{-x/6}$,
  \[ Y(x) \sim D \cos \left[ 5 \cdot 3^{1/4} \left( -\frac{1}{2}x \right)^{5/4} + \phi \right]. \]

†Bender, Fring, & Komijani, J.Phys.A (2014)
Painlevé Transcendental Equations I and II

Painlevé I: $y''(x) = 6y^2(x) + x$

Hyperasymptotic analysis

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  $$Y(x) \sim D \cos \left[5 \cdot 3^{1/4} \left(-\frac{1}{2}x\right)^{5/4} + \phi\right].$$  

- Unstable around the upper asymptote, $y_0(x) \sim \sqrt{-x/6}$,

  $$Y(x) \sim D_1 \exp \left[5 \cdot 3^{1/4} \left(-\frac{1}{2}x\right)^{5/4}\right] + D_2 \exp \left[-5 \cdot 3^{1/4} \left(-\frac{1}{2}x\right)^{5/4}\right].$$

Nonlinear eigenvalue problems†: Choose initial conditions such that $D_1 = 0$.

†Bender, Fring, & Komijani, J.Phys.A (2014)
Two initial conditions $\Rightarrow$ two eigenvalue problems

1. Fixed $y(0) = c$, tune $y'(0) = b_n$. 
Two initial conditions ⇒ two eigenvalue problems

1. Fixed $y(0) = c$, tune $y'(0) = b_n$.

2. Fixed $y'(0) = b$, tune $y(0) = c_n$. 

Painlevé I: $y''(x) = 6y^2(x) + x$
Asymptotic analysis of PII

For simplicity, set $\alpha = 0 \Rightarrow$ three-term equation.

1. Movable singularities: $y''(x) \sim 2y^3(x) \gg x$,

$$y(x) = \pm \frac{1}{(x - x_0)^1} \left[ 1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n \right],$$

where $x_0$ and $a_4$ are two arbitrary constants $\Rightarrow$ general solutions.
Asymptotic analysis of PII

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where $x_0$ and $a_4$ are two arbitrary constants $\Rightarrow$ general solutions

2. Large (negative) $x$ behavior: $2y^3(x) \sim -xy(x) \gg y''(x),$

$$y(x) \sim \pm \sqrt{-\frac{x}{2}}, \quad x \to -\infty.$$ 

Two asymptotes as $x \to -\infty$ & one trivial solution, $y(x) = 0.$
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Two asymptotes as $x \to -\infty$ & one trivial solution, $y(x) = 0.$

3. Airy function-like behavior: $y''(x) \sim xy(x) \gg y^3(x).$
Fluctuations about the asymptotes

Let $Y(x)$ be a small difference between two nearby solutions, $Y(x) \equiv y(x) - y_0(x)$, where $y_0(x)$ and $y(x)$ are solutions of PII.
Fluctuations about the asymptotes

- Let $Y(x)$ be a small difference between two nearby solutions, $Y(x) \equiv y(x) - y_0(x)$, where $y_0(x)$ and $y(x)$ are solutions of PII.
- To the leading order, $Y(x)$ satisfies a linear differential equation,

$$Y''(x) \sim \left[6y_0^2(x) + x\right]Y(x) \sim -2xY(x), \quad x \to -\infty.$$
Fluctuations about the asymptotes

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- Unstable fluctuations about each asymptote,

$$Y(x) \sim D_1 \exp \left[\frac{1}{3} (-2x)^{3/2}\right] + D_2 \exp \left[-\frac{1}{3} (-2x)^{3/2}\right].$$
Fluctuations about the asymptotes

- Let $Y(x)$ be a small difference between two nearby solutions, $Y(x) \equiv y(x) - y_0(x)$, where $y_0(x)$ and $y(x)$ are solutions of PII.
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- Unstable fluctuations about each asymptote,

\[ Y(x) \sim D_1 \exp \left[ \frac{1}{3} (-2x)^{\frac{3}{2}} \right] + D_2 \exp \left[ -\frac{1}{3} (-2x)^{\frac{3}{2}} \right]. \]

- Nonlinear eigenvalue problems: Choose initial conditions such that $D_1 = 0$. 
Different types of solutions

- Two examples of eigenfunctions/separatrix solutions:

\[ y''(x) = 2y^3(x) + xy(x) + \alpha \]

Two examples of other solutions:
Different types of solutions

- Two examples of eigenfunctions/separatrix solutions:

- Two examples of other solutions:
\( \mathcal{PT} \)-symmetric Hamiltonians

- \( \mathcal{PT} \)-symmetric Hamiltonian\(^\ddagger\), \( \hat{H} = \frac{1}{2}\hat{p}^2 + g\hat{x}^2 \text{i} \hat{x}^\varepsilon. \)

\( \mathcal{PT} \)-symmetric Hamiltonians

- \( \mathcal{PT} \)-symmetric Hamiltonian\(^\dagger\), \( \hat{H} = \frac{1}{2} \hat{p}^2 + g \hat{x}^{2k} (i \hat{x})^\epsilon \).
- WKB approximation to the \( n \)th eigenvalues,

\[
E_n \sim \frac{1}{2} (2g)^{2k+\epsilon+2} \left[ \frac{n \sqrt{\pi} \Gamma \left( \frac{3}{2} + \frac{1}{2k+\epsilon} \right)}{\sin \left( \frac{k \pi}{2k+\epsilon} \right) \Gamma \left( 1 + \frac{1}{2k+\epsilon} \right)} \right]^{\frac{2(2k+\epsilon)}{2k+\epsilon+2}}, \quad n \to \infty.
\]

**$\mathcal{PT}$-symmetric Hamiltonians**

- $\mathcal{PT}$-symmetric Hamiltonian‡, $\hat{H} = \frac{1}{2} \hat{p}^2 + g \hat{x}^{2k}(i\hat{x})^\epsilon$.
- WKB approximation to the $n$th eigenvalues,

$$E_n \sim \frac{1}{2} (2g)^{\frac{2}{2k+\epsilon+2}} \left[ \frac{n\sqrt{\pi} \Gamma \left( \frac{3}{2} + \frac{1}{2k+\epsilon} \right)}{\sin \left( \frac{k\pi}{2k+\epsilon} \right) \Gamma \left( 1 + \frac{1}{2k+\epsilon} \right)} \right]^{\frac{2(2k+\epsilon)}{2k+\epsilon+2}}, \quad n \to \infty.$$  

- For example,

  1. $H^{(1)} = \frac{1}{2} \hat{p}^2 + 2i\hat{x}^3$, $E_n^{(1)} \sim 2 \left[ n\sqrt{3\pi} \frac{\Gamma(-\frac{1}{6})}{\Gamma(\frac{1}{3})} \right]^\frac{6}{5}$,

  2. $H^{(2)} = \frac{1}{2} \hat{p}^2 - \frac{1}{2} \hat{x}^4$, $E_n^{(2)} \sim \frac{1}{2} \left[ 3n\sqrt{2\pi} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{4})} \right]^\frac{4}{3}$,

  3. $H^{(3)} = \frac{1}{2} \hat{p}^2 + \frac{1}{2} \hat{x}^4$, $E_n^{(3)} \sim \frac{1}{2} \left[ 3n\sqrt{\pi} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{4})} \right]^\frac{4}{3}$.

Is the $\mathcal{PT}$ symmetry hidden in Painlevé transcendents?

- Multiply $y'(x)$ on PI or PII and then integrate over $x$:

$$\frac{1}{2} [y'(x)]^2 - 2y^3(x) = \frac{1}{2} [y'(0)]^2 - 2y^3(0) + \int_0^x ds \, s y'(s);$$

$$\frac{1}{2} [y'(x)]^2 - \frac{1}{2} y^4(x) = \frac{1}{2} [y'(0)]^2 - \frac{1}{2} y^4(0) + \int_0^x ds \, s y(s) y'(s).$$

Is the $\mathcal{PT}$ symmetry hidden in Painlevé transcendents?

- Multiply $y'(x)$ on PI or PII and then integrate over $x$:

\[
\frac{1}{2} \left[ y'(x) \right]^2 - 2y^3(x) = \frac{1}{2} \left[ y'(0) \right]^2 - 2y^3(0) + \int_0^x ds \, sy'(s);
\]

\[
\frac{1}{2} \left[ y'(x) \right]^2 - \frac{1}{2}y^4(x) = \frac{1}{2} \left[ y'(0) \right]^2 - \frac{1}{2}y^4(0) + \int_0^x ds \, sy(s)y'(s).
\]

- If $\int ds \cdots$ is negligible, and rotate $x$ in the complex plane, we may have:\

\[E^{(1)}_n \sim \frac{1}{2} \left[ y'(0) \right]^2 - 2y^3(0),\]

\[E^{(2)}_n \sim \frac{1}{2} \left[ y'(0) \right]^2 - \frac{1}{2}y^4(0),\]

\[E^{(3)}_n \sim \frac{1}{2} \left[ y'(0) \right]^2 + \frac{1}{2}y^4(0).\]

\[\S\text{Bender & Komijani, J.Phys.A (2015)}\]
Painlevé transcendents are $\mathcal{PT}$-symmetric!

**Painlevé I**

- $y_n'(0) \sim \sqrt{2E_n^{(1)}} \sim 2 \left[ n \sqrt{3\pi} \frac{\Gamma(\frac{11}{6})}{\Gamma(\frac{1}{3})} \right]^{\frac{3}{5}} \approx 2.092\,146\,74 \, n^{\frac{3}{5}}$.

- $y_n(0) \sim -\left[ \frac{1}{2} E_n^{(1)} \right]^{\frac{1}{3}} \sim - \left[ n \sqrt{3\pi} \frac{\Gamma(\frac{11}{6})}{\Gamma(\frac{1}{3})} \right]^{\frac{2}{5}} \approx -1.030\,484\,4 \, n^{\frac{2}{5}}$. 
**Painlevé transcendents are $\mathcal{PT}$-symmetric!**

**Painlevé I**

- $y'_n(0) \sim \sqrt{2E_n^{(1)}} \sim 2 \left[ n\sqrt{3\pi} \frac{\Gamma(\frac{11}{6})}{\Gamma(\frac{1}{3})} \right]^{\frac{3}{5}} \approx 2.092\,146\,74\,n^{\frac{3}{5}}$

- $y_n(0) \sim \left[ \frac{1}{2} E_n^{(1)} \right]^{\frac{1}{3}} \sim -\left[ n\sqrt{3\pi} \frac{\Gamma(\frac{11}{6})}{\Gamma(\frac{1}{3})} \right]^{\frac{2}{5}} \approx -1.030\,484\,4\,n^{\frac{2}{5}}$

**Painlevé II**

- $y'_n(0) \sim \sqrt{2E_n^{(2)}} \sim \left[ 3n\sqrt{2\pi} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right]^{\frac{2}{3}} \approx 1.862\,412\,8\,n^{\frac{2}{3}}$

- $y_n(0) \sim \left[ 2E_n^{(3)} \right]^{\frac{1}{4}} \sim \left[ 3n\sqrt{\pi} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right]^{\frac{1}{3}} \approx 1.215\,811\,65\,n^{\frac{1}{3}}$
Outline

1. Linear Eigenvalue Problems

2. Painlevé Transcendental Equations I and II

3. Super Painlevé Equations
   - A new class of nonlinear differential equations
   - “Regular” eigenvalue problems
   - “Peculiar” eigenvalue problems
   - “Bizarre” eigenvalue problems
   - General super Painlevé equations

4. Conclusion and Future Work
Super Painlevé equations

- \( SP(M,N) \):

\[
y''(x) = \frac{2(M + 1)}{(M - 1)^2} y^M(x) + xy^N(x).
\]
Super Painlevé equations

- **SP(M,N):**

  \[
  y''(x) = \frac{2(M + 1)}{(M - 1)^2} y^M(x) + xy^N(x).
  \]

- Movable singularities:

  \[
  y(x) = \begin{cases} 
  \frac{1}{(x - x_0)^{\frac{2}{M-1}}} \left[ 1 + \sum_{n=1}^{\infty} a_n (x - x_0)^{\frac{n}{M-1}} \right], & M = 2k; \\
  \pm \frac{1}{(x - x_0)^{\frac{1}{k}}} \left[ 1 + \sum_{n=1}^{\infty} a_n (x - x_0)^{\frac{n}{k}} \right], & M = 2k + 1.
  \end{cases}
  \]

  For even \( M = 2k \), \( a_2(M+1) \) is arbitrary.
  For odd \( M = 2k + 1 \), \( a_2(k+1) \) is arbitrary. Only odd \( k \) may have real eigensolutions. SP(3,1) is PII with \( \alpha = 0 \).
Super Painlevé equations

- **SP(M,N):**
  \[ y''(x) = \frac{2(M + 1)}{(M - 1)^2} y^M(x) + x y^N(x). \]
  
- **Movable singularities:**
  \[ y(x) = \begin{cases} 
  \frac{1}{(x - x_0)^{M-1}} \left[ 1 + \sum_{n=1}^{\infty} a_n (x - x_0)^{\frac{n}{M-1}} \right], & M = 2k; \\
  \pm \frac{1}{(x - x_0)^{\frac{1}{k}}} \left[ 1 + \sum_{n=1}^{\infty} a_n (x - x_0)^{\frac{n}{k}} \right], & M = 2k + 1. 
\]

- For even \( M = 2k \), \( a_{2(M+1)} \) is arbitrary. SP(2,0) is PI.
Super Painlevé equations

- **SP(M,N):**

\[
y''(x) = \frac{2(M + 1)}{(M - 1)^2} y^M(x) + xy^N(x).
\]

- **Movable singularities:**

\[
y(x) = \frac{1}{(x - x_0)^{\frac{M-1}{2}}} \left[1 + \sum_{n=1}^{\infty} a_n (x - x_0)^{\frac{n}{M-1}}\right], \quad M = 2k;
\]

\[
y(x) = \pm \frac{1}{(x - x_0)^{\frac{1}{k}}} \left[1 + \sum_{n=1}^{\infty} a_n (x - x_0)^{\frac{n}{k}}\right], \quad M = 2k + 1.
\]

- For even \( M = 2k \), \( a_{2(M+1)} \) is arbitrary. SP(2,0) is PI.
- For odd \( M = 2k + 1 \), \( a_{2(k+1)} \) is arbitrary. Only odd \( k \) may have real eigensolutions. SP(3,1) is PII with \( \alpha = 0 \).
Super Painlevé Equations

“Regular” eigenvalue problems

\[ SP(4,0): \quad y''(x) = \frac{10}{9} y^4(x) + x \]

\[ y'_n(0) \sim 1.1102n^{\frac{5}{11}} \]

\[ y'_n(0) \sim -1.11n^{\frac{5}{11}} \]
SP(4,0): $y''(x) = \frac{10}{9} y^4(x) + x$

- $y'_n(0) \sim 1.1102n^{\frac{5}{11}}$
- $y'_n(0) \sim -1.11n^{\frac{5}{11}}$
- $y_n(0) \sim 1.80547n^{\frac{2}{11}}$
- $y_n(0) \sim -1.226n^{\frac{2}{11}}$
Super Painlevé Equations

“Regular” eigenvalue problems

$\text{SP}(4,1): \quad y''(x) = \frac{10}{9} y^4(x) + x y(x)$

$y_n'(0) \sim 2.1336 n^{\frac{5}{9}}$

$y_n'(0) \sim -2.1336 n^{\frac{5}{9}}$
SP(4,1): \( y''(x) = \frac{10}{9} y^4(x) + x y(x) \)

- \( y'_n(0) \sim 2.1336n^{\frac{5}{9}} \)
- \( y'_n(0) \sim -2.1336n^{\frac{5}{9}} \)

No solutions for \( y_n(0) > 0 \)

- \( y_n(0) \sim -1.59255n^{\frac{2}{9}} \)
Super Painlevé Equations

“Regular” eigenvalue problems

SP(4,2): \( y''(x) = \frac{10}{9} y^4(x) + xy^2(x) \)

\[
\begin{align*}
    y'_n(0) & \sim 2.9996n^{\frac{5}{7}} \\
    y'_n(0) & \sim -2.9996n^{\frac{5}{7}}
\end{align*}
\]
Super Painlevé Equations

"Regular" eigenvalue problems

SP(4,2): \( \ddot{y}(x) = \frac{10}{9} y^4(x) + xy^2(x) \)

\[
\begin{align*}
    y'_n(0) & \sim 2.9996n^{\frac{5}{7}} \\
    y'_n(0) & \sim -2.9996n^{\frac{5}{7}} \\
    y_n(0) & \sim 1.098102n^{\frac{2}{7}} \\
    y_n(0) & \sim -1.82502n^{\frac{2}{7}}
\end{align*}
\]
Super Painlevé Equations

“Regular” eigenvalue problems

SP(M,M-m): \( y''(x) = A \left[ y^M(x) + Bxy^{M-m}(x) \right] \)

- Scaling: \( y = \alpha Y, \quad x = aX. \)
Super Painlevé Equations

“Regular” eigenvalue problems

\[ \text{SP}(M,M-m): \quad y''(x) = A \left[ y^M(x) + B x y^{M-m}(x) \right] \]

 Scaling: \( y = \alpha Y, \ x = aX. \)
\[ Y' = a\alpha^{-1} y', \ Y'' = a^2\alpha^{M-1} A \left[ Y^M + a\alpha^{-m} B X Y^{M-m} \right]. \]
Super Painlevé Equations

**SP(M,M-m):** \[ y''(x) = A \left[ y^M(x) + Bxy^{M-m}(x) \right] \]

- **Scaling:** \( y = \alpha Y, \ x = aX \).

- \( Y' = a\alpha^{-1}y', \ Y'' = a^2\alpha^{M-1}A \left[ Y^M + a\alpha^{-m}BXY^{M-m} \right] \).

- If \( y'_{\nu}(0) = C_nA^{\mu}B^{\nu} \), then

\[
Y'_{\nu}(0) = C_n \left( a^2\alpha^{M-1}A \right)^{\mu} \left( a\alpha^{-m}B \right)^{\nu} = a^{2\mu+\nu}\alpha^{(M-1)\mu-m\nu}y'_{\nu}(0).
\]
Super Painlevé Equations

"Regular" eigenvalue problems

\[ \text{SP}(M,M-m): \ y''(x) = A \left[ y^M(x) + Bx y^{M-m}(x) \right] \]

- Scaling: \( y = \alpha Y, \ x = aX \).
- \( Y' = a\alpha^{-1}y', \ Y'' = a^2\alpha^{M-1}A \left[ Y^M + a\alpha^{-m}BXY^{M-m} \right] \).
- If \( y'_n(0) = C_n A^\mu B^\nu \), then
  \[
  Y'_n(0) = C_n \left( a^2\alpha^{M-1}A \right)^\mu \left( a\alpha^{-m}B \right)^\nu = a^{2\mu+\nu} \alpha^{(M-1)\mu-m\nu} y'_n(0).
  \]

\[
\begin{align*}
2\mu + \nu &= 1 \\
(M-1)\mu - m\nu &= -1 \\
\end{align*}
\]

\[
\begin{align*}
\mu &= \frac{m-1}{M+2m-1} \\
\nu &= \frac{M+1}{M+2m-1}
\end{align*}
\]
Super Painlevé Equations

“Regular” eigenvalue problems

\[ \text{SP}(M, M-m): \quad y''(x) = A \left[ y^M(x) + B x y^{M-m}(x) \right] \]

- Scaling: \( y = \alpha Y, \quad x = aX \).
- \( Y' = a\alpha^{-1} y', \quad Y'' = a^2 \alpha^{M-1} A \left[ Y^M + a\alpha^{-m} B X Y^{M-m} \right] \).
- If \( y'_n(0) = C_n A^\mu B^\nu \), then
  \[
  Y'_n(0) = C_n \left( a^2 \alpha^{M-1} A \right)^\mu \left( a\alpha^{-m} B \right)^\nu = a^{2\mu+\nu} \alpha^{(M-1)\mu-m\nu} y'_n(0).
  \]

\[
\begin{align*}
2\mu + \nu &= 1 \\
(M - 1)\mu - m\nu &= -1
\end{align*}
\]

\[
\Rightarrow \quad \begin{cases}
\mu = \frac{m-1}{M+2m-1} \\
\nu = \frac{M+1}{M+2m-1}
\end{cases}
\]

- We found numerically that \( C_n \sim C_n^{\frac{M+1}{M+2m-1}}, \quad n \to \infty \).
Super Painlevé Equations

```
SP(M,M-m): y''(x) = A [y^M(x) + Bx y^{M-m}(x)]
```

- Scaling: \( y = \alpha Y, \ x = aX. \)
- \( Y' = a\alpha^{-1}y', \ Y'' = a^2 \alpha^{M-1} A \left[ Y^M + a\alpha^{-m} BXY^{M-m} \right]. \)
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- Put them all together, we get
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  \[ y'_n(0) \sim CA^{\frac{m-1}{M+2m-1}}(Bn)^{\frac{M+1}{M+2m-1}}. \]

- Similarly,
  
  \[ y_n(0) \sim DA^{-\frac{1}{M+2m-1}}(Bn)^{\frac{2}{M+2m-1}}. \]
Super Painlevé Equations

“Peculiar” eigenvalue problems

\[ SP(4,2): \quad y''(x) = \frac{10}{9} y^4(x) + x y^2(x) \]

- There are rich features for super Painlevé equations.
Super Painlevé Equations

“Peculiar” eigenvalue problems

$\text{SP}(4,2): \ y''(x) = \frac{10}{9} y^4(x) + xy^2(x)$

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- Infinite “peculiar” eigensolutions near every “regular” eigensolution.
Super Painlevé Equations  

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Super Painlevé Equations

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### Graphs

- **Graph 1:**
  - Function plot showing oscillations and asymptotic behavior.

- **Graph 2:**
  - Plot of eigenvalues \( \lambda_n \) vs. \( n \)

- **Equations:**
  \[ c_n = 1.427047 - \lambda_n \]
  \[ \lambda_n \sim 4.1789 \times 10^{-4.02244(n-1)} \]
SP(6,3): \( y''(x) = \frac{14}{25} y^6(x) + x y^3(x) \)

For some super Painlevé equations, there is yet another type of eigensolutions. For example,
Super Painlevé Equations  "Bizarre" eigenvalue problems

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SP(6,3) in the positive \( x \) domain. The eigenvalues seem not following any power law or exponential law for large \( n \).
More super Painlevé equations

- General form of SP equation

\[ y''(x) = Ay^M(x) + xP[y(x)] + Q[y(x)], \]
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where

\[ P[y] = C_{M-2}y^{M-2} + C_{M-3}y^{M-3} + \cdots + C_1y + C_0, \]

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- For odd \( M \), things get complicated.
General super Painlevé equations with odd $M$

- **SP3:** $C_0 = 0 \Rightarrow$ general form of PII:

$$y''(x) = 2y^3(x) + xy(x) + D_0.$$
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$$y''(x) = \frac{3}{4}y^5(x) + x \left[ C_2 y^2(x) + C_0 \right]$$

$$+ D_3 y^3(x) + D_2 y^2(x) + D_1 y(x) + D_0.$$
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- **SP11:** there are three cases.
Outline

1. Linear Eigenvalue Problems
2. Painlevé Transcendental Equations I and II
3. Super Painlevé Equations
4. Conclusion and Future Work
Concluding remarks

- Nonlinear eigenvalue problems are linked to $\mathcal{PT}$-symmetric Hamiltonians.
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- Two new types of nonlinear eigenvalue problems are proposed.
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- Nonlinear eigenvalue problems are linked to $\mathcal{PT}$-symmetric Hamiltonians.
- Painlevé transcendents are generalized to an infinite class of nonlinear equations with only algebraic movable singularities.
- Separatrix solutions are found to satisfy discrete initial conditions, $\Rightarrow$ eigenvalues.
- Two new types of nonlinear eigenvalue problems are proposed.
- A lot more to do · · ·
  - How to derive the prefactors in the large $n$ behaviors in the “regular” eigenvalues of the super Painlevé equations?
  - How to derive the exponential behavior of the “peculiar” eigenvalues?
  - What’s the large $n$ behavior for the “bizarre” eigenvalues?
  - Any other significance of these equations?