Eigenvalue inequalities for the Laplacian with mixed boundary conditions

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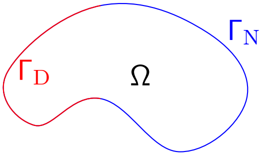
Background and motivation

Dirichlet, Neumann, and mixed eigenvalues

A bounded domain $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ with sufficiently regular boundary $\partial\Omega$.



ν – unit normal vector on $\partial\Omega$

$$\partial\Omega = \overline{\Gamma_{\rm D} \cup \Gamma_{\rm N}}$$

$$\Gamma_{\rm D} \cap \Gamma_{\rm N} = \varnothing, \ \Gamma := \Gamma_{\rm D}$$

Mixed eigenvalues

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_{\rm D}, \\ \nu \cdot \nabla u = 0, & \text{on } \Gamma_{\rm N}. \end{cases} \quad \Longrightarrow \quad 0 \leq \lambda_1^\Gamma < \lambda_2^\Gamma \leq \lambda_3^\Gamma \leq \dots$$

Dirichlet and Neumann eigenvalues

$$\lambda_k := \lambda_k^{\partial\Omega} \ \ (\text{Dir.}) \qquad \text{and} \qquad \mu_k := \lambda_k^{\varnothing} \ \ (\text{Neu.})$$

Eigenvalues inequalities

Comparison of Dirichlet and Neumann eigenvalues

$$\mu_k \leq \lambda_k \quad \forall\, k \in \mathbb{N} \text{ (trivial)}$$

$$\mu_2 < \lambda_1 \quad \text{for } \Omega \subset \mathbb{R}^2 \text{ (Pólya-52)}$$

$$\mu_{k+2} < \lambda_k \quad \forall\, k \in \mathbb{N} \text{ and convex, } C^2\text{-smooth } \Omega \subset \mathbb{R}^2 \text{ (Payne-55)}$$

$$\mu_{k+d} \leq \lambda_k \quad \forall\, k \in \mathbb{N} \text{ and convex } \Omega \text{ (Levine-Weinberger-86)}$$

$$\mu_{k+1} \leq \lambda_k \quad \forall\, k \in \mathbb{N} \text{ and } C^1\text{-smooth } \Omega \text{ (Friedlander-91)}$$

$$\mu_{k+1} < \lambda_k \quad \forall\, k \in \mathbb{N} \text{ and Lipschitz } \Omega \text{ (Filonov-04)}$$

$\mu_k \leq \lambda_k^\Gamma \leq \lambda_k$, $\forall k \in \mathbb{N}$ (trivial).

Our ultimate goal

To generalize inequalities of Pólya, Payne, Levine-Weinberger, Friedlander, and Filonov for mixed eigenvalues.

Some applications of mixed eigenvalue problem

Nodal domains

Mixed eigenvalues are used in the analysis of nodal domains for Neumann eigenfunctions.

Mixed EVs arise in the proof of the **hot spot conjecture** (J. Rauch)

The hottest point on an insulated plate Ω moves towards $\partial\Omega$ as $t \nearrow \infty$.

A reformulation: Global extrema for the 2^{nd} Neumann eigenfunction are not attained inside Ω.

Mixed EVs are used in the construction of **isospectral domains**

Answering negatively to 'Can one hear the shape of the drum?' (Sunada-85, Gordon-Webb-Wolpert-92, Arendt-terElst-Kennedy-13).

An operator-theoretic interpretation of mixed eigenvalues

$H^1_{0,\Gamma}(\Omega) := \{u \in H^1(\Omega)\colon u|_\Gamma = 0\}$ – Sobolev-type space.

Non-negative quadratic form

$$H^1_{0,\Gamma}(\Omega) \ni u \mapsto \mathfrak{h}_\Gamma[u] := \int_\Omega |\nabla u(x)|^2 \mathrm{d}x.$$

Proposition

The form $\mathfrak{h}_\Gamma$ is closed, densely defined, and symmetric in $L^2(\Omega)$.

1st repr. thm. associates to $\mathfrak{h}_\Gamma$ the (!) self-adjoint operator in $L^2(\Omega)$

$-\Delta_\Gamma u = -\Delta u$, $\mathrm{dom}\,(-\Delta_\Gamma) = \{u \in H^1_{0,\Gamma}(\Omega)\colon \Delta u \in L^2(\Omega), \nu \cdot \nabla u = 0 \text{ on } \Gamma_\mathrm{N}\}$

Interpretation

Mixed eigenvalues are naturally interpreted as the eigenvalues of $-\Delta_\Gamma$.

Main results

Preliminaries

Let $\Omega \subset \mathbb{R}^d$ be a bounded, connected Lipschitz domain.

Rademacher's theorem

For almost all $x \in \partial\Omega$, there exists a unit normal vector $\nu(x)$.

A linear subspace of $\mathbb{R}^d$ associated to $x \in \partial\Omega$

$$\mathcal{T}_x = \left\{\tau \in \mathbb{R}^d \colon \tau \cdot \nu(x) = 0\right\} \subset \mathbb{R}^d.$$

$\mathcal{T}_x$ is a hyperplane in $\mathbb{R}^d$ which contains the origin and consists of vectors orthogonal to $\nu(x)$.

A linear subspace of $\mathbb{R}^d$ associated to $\Sigma \subset \partial\Omega$

$$\mathcal{S}(\Sigma) = \cap_{x\in\Sigma}\mathcal{T}_x.$$

$\dim \mathcal{S}(\Sigma) \in \{0, 1, 2 \dots, d-1\}$.

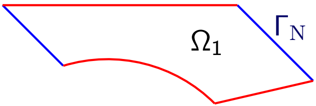
Mixed and Neumann eigenvalues

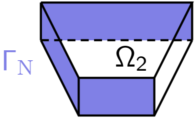
Theorem A (VL-Rohleder-17)

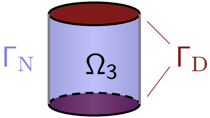
Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then

$$\dim \mathcal{S}(\Gamma_\mathrm{N}) \geq 1 \quad \Longrightarrow \quad \mu_{k+1} \leq \lambda_k^\Gamma \quad \textit{for all} \quad k \in \mathbb{N}.$$

In the Friedlander-Filonov inequality Neumann boundary condition can be left on a part of $\partial\Omega$.





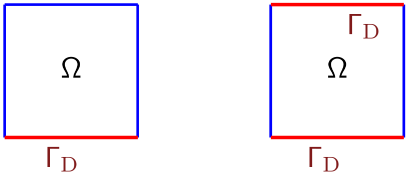


In all these examples $\dim \mathcal{S}(\Gamma_\mathrm{N}) = 1$.

Necessity of $\dim \mathcal{S}(\Gamma_N) \geq 1$

$\Omega = [0,\pi]^2 \subset \mathbb{R}^2$

- $\Gamma_D = (0,\pi) \times \{0\}$: $\dim \mathcal{S}(\Gamma_N) = 0$, $\mu_2 = 1 > 1/4 = \lambda_1^\Gamma$ (ineq. fails).
- $\Gamma_D = (0,\pi) \times \{0,\pi\}$: $\dim \mathcal{S}(\Gamma_N) = 1$, $\mu_2 = 1 = \lambda_1^\Gamma$ (non-strict).



The strict inequality holds under an extra assumption.

Proposition

$\Gamma_N \subset \Sigma \subset \partial\Omega$, $|\Sigma \setminus \Gamma_N| > 0$, $\dim \mathcal{S}(\Sigma) \geq 1 \Rightarrow \mu_{k+1} < \lambda_k^\Gamma$, $\forall k \in \mathbb{N}$.

A combination of Thm. A and a unique continuation argument.

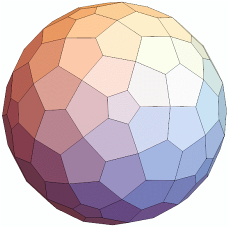
Polyhedra

Definition

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded, connected Lipschitz domain.

(i) *$d = 2$: Ω is polyhedral if $\partial\Omega$ is the union of finitely many line segments.*

(ii) *$d \geq 3$: Ω is polyhedral if for any hyperplane $H \subset \mathbb{R}^d$ the intersection $H \cap \Omega$ is either polyhedral in $\mathbb{R}^{d-1}$ or $\varnothing$.*



Mixed and Dirichlet eigenvalues for convex polyhedra

Theorem B (VL-Rohleder-17)

Let $\Omega \subset \mathbb{R}^d$ be a convex, polyhedral domain. Then

$$\dim \mathcal{S}(\Gamma_{\rm D}) = \ell \geq 1 \quad \Longrightarrow \quad \lambda^{\Gamma}_{k+\ell} \leq \lambda_k \quad \textit{for all } k \in \mathbb{N}.$$

Ω_1: $\dim \mathcal{S}(\Gamma_{\rm D}) = 2 \Rightarrow \ \lambda^{\Gamma}_{k+2} \leq \lambda_k, \forall k \in \mathbb{N}$.

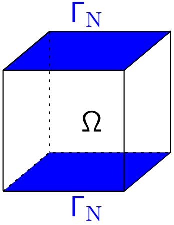
Ω_2: $\dim \mathcal{S}(\Gamma_{\rm D}) = 1 \Rightarrow \ \lambda^{\Gamma}_{k+1} \leq \lambda_k, \forall k \in \mathbb{N}$.

Necessity of $\dim \mathcal{S}(\Gamma_{\rm D}) \geq \ell$

$\Omega = [0,\pi]^3 \subset \mathbb{R}^3$ and $\Gamma_{\rm N} = [0,\pi]^2 \times \{0,\pi\}$, $\dim \mathcal{S}(\Gamma_{\rm D}) = 1$

$$\lambda_3^\Gamma = 5 > 3 = \lambda_1 \qquad \text{(inequality fails for } \ell = 2\text{)}$$

$$\lambda_2^\Gamma = 3 = \lambda_1 \qquad \text{(strict inequality fails for } \ell = 1\text{)}$$



Proposition

$\Gamma_{\rm D} \subset \Sigma \subset \partial\Omega$, $|\Sigma \setminus \Gamma_{\rm D}| > 0$, $\dim \mathcal{S}(\Sigma) = \ell \geq 1 \;\Rightarrow \lambda_{k+\ell}^\Gamma < \lambda_k$, $\forall k \in \mathbb{N}$.

Thm. B + unique continuation argument.

Ideas of the proofs and discussion

Ideas of the proofs

Thm. A: $\dim \mathcal{S}(\Gamma_{\rm N}) \geq 1 \implies \mu_{k+1} \leq \lambda_k^\Gamma$, $\forall\, k \in \mathbb{N}$.

(i) Minmax for $-\Delta_{\rm N}$ with EFs $\{u_j^\Gamma\}_{j=1}^k$ of $-\Delta_\Gamma$ & $x \mapsto e^{i\omega\cdot x}$, $\omega \in \mathbb{R}^d$.

(ii) Filonov-04: only the length of ω fixed.[1]

(iii) New idea: $\omega \perp \nu(x)$ for all $x \in \Gamma_{\rm N}$, which exists *iff* $\dim \mathcal{S}(\Gamma_{\rm N}) \geq 1$.

Thm. B: $\dim \mathcal{S}(\Gamma_{\rm D}) = \ell \Rightarrow \lambda_{k+\ell}^\Gamma \leq \lambda_k$, $\forall k \in \mathbb{N}$ (convex polyhedra)

(i) Minmax for $-\Delta_\Gamma$ with the EFs $\{u_j^{\rm D}\}_{j=1}^k$ of $-\Delta_{\rm D}$ & part. deriv. of u_k.

(ii) Levine-Weinberger-86: part. deriv. of $u_k^{\rm D}$ in d orth. directions.[2]

(iii) 1st new idea: $v_i \cdot \nabla u_k^{\rm D}$; $\{v_i\}_{i=1}^\ell$ is an orthonormal basis of $\mathcal{S}(\Gamma_{\rm D})$.

(iv) 2nd new idea: Grisvard's integration by parts (convex polyhedra).

[1] N. Filonov, *St. Petersburg Math. J.*, **16**, 2004.

[2] H. Levine and H. Weinberger, *Arch. Ration. Mech. Anal.*, **94**, 1986.

Further challenges

On Theorem A

(i) A generalisation for unbounded Ω with $|\Omega| < \infty$.

(ii) What are extra conditions on Ω and Γ to replace $\mu_{k+1} \le \lambda_k^\Gamma$ by $\mu_{k+\ell} \le \lambda_k^\Gamma$?

On Theorem B

Does a similar result hold for Ω other than convex polyhedra?

$\lambda_{k+\ell}^{\Gamma_1} \le \lambda_k^{\Gamma_2}$ for $\Gamma_1 \subset \Gamma_2$?

Summary

Mixed and Neumann eigenvalues

$\mu_{k+1} \le \lambda_k^\Gamma$, $\forall\, k \in \mathbb{N}$, is shown assuming that $\Gamma_{\rm N}$ is 'small': $\dim \mathcal{S}(\Gamma_{\rm N}) \ge 1$.

Mixed and Dirichlet eigenvalues

$\lambda_{k+\ell}^\Gamma \le \lambda_k$, $\forall\, k \in \mathbb{N}$, is shown assuming that $\Gamma_{\rm D}$ is 'small': $\dim \mathcal{S}(\Gamma_{\rm D}) \ge \ell$.

Some highlights

(i) Smallness of $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$ is understood in somewhat algebraic sense.

(ii) Strict eigenvalue inequalities require extra conditions.

(iii) Solvable examples show necessity of assumptions.

Reference

- V. L. and J. Rohleder, *Eigenvalue inequalities for the Laplacian with mixed boundary conditions*, J. Differential Equations **263** (2017), 491–508.

Thank you for your attention!