Surprising spectra of $\mathcal{PT}$-symmetric point interactions

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\textbf{Why point interactions?}

- solvable models with both continuous and point spectra
- explicit formulas for metric operators
  
  2005 Albeverio, Kuzhel, 2006 Krejčiřík, Bíla, Znojil, 2008 Siegl

- resolvent criterion for similarity to the self-adjoint operator is applicable in a straightforward way
  
  2005 Albeverio, Kuzhel

real spectrum is not sufficient for similarity to the self-adjoint operator (quasi-Hermiticity)
**PT-symmetric point interactions**

**Definitions of operators**

- line $L^2(\mathbb{R})$ or finite interval (circle) $L^2(a, b)$
- $H = -\frac{d^2}{dx^2}$
- $\text{Dom}(H) = AC^1 + \text{boundary conditions at } x = 0 \text{ or at } x = a, b$
\( \mathcal{P}\mathcal{T} \)-symmetric point interactions

\( \mathcal{P}\mathcal{T} \)-symmetric boundary conditions

\[
\begin{pmatrix}
\psi(0^+) \\
\psi'(0^+)
\end{pmatrix}
= B
\begin{pmatrix}
\psi(0^-) \\
\psi'(0^-)
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
\sqrt{1+bce^{i\phi}} & b \\
c & \sqrt{1+bce^{-i\phi}}
\end{pmatrix}, \quad b \geq 0, c \geq -1/b, \quad \phi \in (-\pi, \pi]
\]
System on a line - interaction at $x = 0$

**Symmetries**

- **$\mathcal{PT}$-symmetry:** $\mathcal{PT}H\psi = H\mathcal{PT}\psi$, $\forall \psi \in \text{Dom}(H)$
- **$\mathcal{P}$-pseudo-Hermiticity:** $H^* = \mathcal{P}Hp\mathcal{P}$
- **$\mathcal{T}$-self-adjointness:** $H^* = \mathcal{T}H\mathcal{T}$
- **$\mathcal{T}$-complex conjugation, $\mathcal{P}$-parity**

**Spectrum**

- residual part is empty $\sigma_r(H) = \emptyset$ 2008 Borisov, Krejčiřík
- continuous spectrum $\sigma_c(H) = [0, \infty)$
- $b \neq 0, c \neq 0$ point spectrum - at most two eigenvalues
  - real if $bc \sin^2 \phi \leq \cos^2 \phi$ or $bc \sin^2 \phi \geq \cos^2 \phi$ and $\cos \phi \geq 0$
2002 Albeverio, Fei, Kurasov
Special case $b = 0, c = 0$

**Definition of operator**
- $L^2(\mathbb{R})$
- $H_\phi = -\frac{d^2}{dx^2}$
- $\text{Dom}(H_\phi) = AC^1(\mathbb{R})$
- $\psi(0+) = e^{i\phi}\psi(0-)$
- $\psi'(0+) = e^{-i\phi}\psi'(0-)$,

**Symmetries**
- $\mathcal{P}\mathcal{T}H_\phi\psi = H_\phi\mathcal{P}\mathcal{T}\psi$
- $H_\phi^* = H_{-\phi}$
- $H_\phi^* = \mathcal{P}H_\phi\mathcal{P}$
- $H_\phi^* = \mathcal{T}H_\phi\mathcal{T}$
- $H_\phi$ is closed

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Surprising spectra of $\mathcal{P}\mathcal{T}$ PI
Special case $b = 0, c = 0$

Boundary conditions

$$
\psi(0+) = e^{i\phi}\psi(0-)
$$
$$
\psi'(0+) = e^{-i\phi}\psi'(0-)
$$

Special cases

- $\phi = 0$ - self-adjoint operator, no interaction
  $$
  \psi(0+) = \psi(0-), \quad \psi'(0+) = \psi'(0-)
  $$
- $\phi \neq \pm \pi/2$ - continuous spectrum $[0, \infty)$, no eigenvalues, quasi-Hermitian
- $\phi = \pm \pi/2$ - SURPRISING CASE
  $$
  \psi(0+) = \pm i\psi(0-), \quad \psi'(0+) = \mp i\psi'(0-)
  $$
Special case $b = 0, c = 0$

**Boundary conditions**

\[
\psi(0^+) = e^{i\phi}\psi(0^-)
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**Boundary conditions**

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Special case $b = 0, c = 0, \phi \neq \pm\pi/2$

**Quasi-Hermiticity**

- $H_\phi$ is quasi-Hermitian:
  \[
  \Theta H_\phi^* = H_\phi \Theta, \quad \Theta, \Theta^{-1} \in \mathcal{B}(\mathcal{H}), \quad \Theta > 0
  \]
- \[
  \Theta = I - i \sin \phi P_{\text{sign}} \mathcal{P}
  \]
  \[
  (P_{\text{sign}} f)(x) = \text{sign}(x) f(x), \quad \mathcal{P}-\text{parity}
  \]

**Metric operator $\Theta$**

- spectrum - only two eigenvalues $1 - \sin \phi, 1 + \sin \phi$
- $\Theta > 0, \Theta^{-1} \in \mathcal{B}(\mathcal{H})$
- $\Theta H_\phi^* = H_\phi \Theta$ is valid
- $\Theta$ is not invertible if $\phi = \pm\pi/2$ !
Special case $b = 0, c = 0, \phi \neq \pm \pi/2$

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  \[ (P_{\text{sign}} f)(x) = \text{sign} x f(x), \mathcal{P}-\text{parity} \]

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Special case \( b = 0, \ c = 0, \ \phi \neq \pm \pi/2 \)

**Construction of \( \Theta \)** 2008 Siegl

- finite interval \((-l, l)\) - interaction at \( x = 0 \), Dirichlet BC at \( \pm l \)
- discrete spectrum \( \lambda_n = \left( \frac{n\pi}{2l} \right)^2, \ n \in \mathbb{N}_0 \)
- eigenfunctions
  
  \[
  \psi_{2n}(x) = (e^{-i\phi} \vartheta(x) + \vartheta(-x)) \sin \frac{n\pi}{l} x \\
  \psi_{2n+1}(x) = (e^{i\phi} \vartheta(x) - \vartheta(-x)) \cos \frac{(2n+1)\pi}{2l} x
  \]
  
- \( \Theta = \text{s-lim}_{N \to \infty} \sum_{j=1}^{N} c_j \langle \phi_j, \cdot \rangle \phi_j = I - i \sin \phi P_{\text{sign}} \mathcal{P} \)
- \( \phi_n = \mathcal{P} \psi_n \) eigenfunctions of \( H^* \)
- limit \( l \to \infty \)
Surprising case $b = 0, c = 0, \phi = \pi/2$

Properties of $H_{\pi/2}$

- $\psi(0+) = i\psi(0-)$, $\psi'(0+) = -i\psi'(0-)$
- $H_{\pi/2}$ is $\mathcal{PT}$-symmetric, $\mathcal{P}$-pseudo-Hermitian, $\mathcal{T}$-self-adjoint
- $H_{\pi/2}^* = H_{-\pi/2}$, $H_{\pi/2}$ is closed
- $\Theta H_{\pi/2}^* = H_{\pi/2} \Theta$
- $\Theta = I - iP_{\text{sign}} \mathcal{P}$, $\Theta \geq 0$, $\Theta$ is not invertible!

Spectrum

- residual spectrum is empty
- continuous spectrum $[0, \infty)$
- point spectrum $\mathbb{C} \setminus [0, \infty)$!
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### Properties of \( H_{\pi/2} \)
- \( \psi(0^+) = i\psi(0^-), \psi'(0^+) = -i\psi'(0^-) \)
- \( H_{\pi/2} \) is \( \mathcal{PT} \)-symmetric, \( \mathcal{P} \)-pseudo-Hermitian, \( \mathcal{T} \)-self-adjoint
- \( H_{\pi/2}^* = H_{-\pi/2}, H_{\pi/2} \) is closed
- \( \Theta H_{\pi/2}^* = H_{\pi/2} \Theta \)
- \( \Theta = I - iP_{\text{sign}}\mathcal{P}, \Theta \geq 0, \Theta \) is not invertible!

### Spectrum
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**Spectrum**

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Surprising case $b = 0, c = 0, \phi = \pi/2$

Eigenfunctions of $H_{\pi/2}$

\[
\psi_k(x) = \begin{cases} 
  e^{kx}, & x < 0, \\
  ie^{-kx}, & x > 0,
\end{cases}
\]

\[
\varphi_k(x) = \begin{cases} 
  e^{-kx}, & x < 0, \\
  ie^{kx}, & x > 0,
\end{cases}
\]

\[
\zeta_k(x) = \begin{cases} 
  e^{-ikx}, & x < 0, \\
  ie^{ikx}, & x > 0.
\end{cases}
\]

$\psi_k \in L^2(\mathbb{R})$ for $\text{Re} \, k > 0$, $\varphi_k \in L^2(\mathbb{R})$ for $\text{Re} \, k < 0$, $\zeta_k \in L^2(\mathbb{R})$ for $\text{Re} \, k = 0$ and $\text{Im} \, k > 0$. 
Models on finite interval

Models on a finite interval \((-l, l)\)

- \(L^2(-l, l), H = -\frac{d^2}{dx^2}\)
- \(\text{Dom}(H) = AC^1(-l, l)\)
- 2 interactions - at \(x = 0\) and \(x = \pm l\) - 2 BC
- at \(x = 0\) - \(\mathcal{PT}\)-symmetric interaction \(b = 0, c = 0\)
- at \(x = \pm l\) - both self-adjoint and \(\mathcal{PT}\)-symmetric interactions
Compact resolvent guaranteed?

Theorem (Kato)

Let $T_1, T_2 \in \mathcal{C}(\mathcal{H})$ have non-empty resolvent sets. Let $T_1, T_2$ be extensions of a common operator $T_0$, with order of extension for $T_1$ being finite. Then $T_1$ has compact resolvent if and only if $T_2$ has compact resolvent.
\( \mathcal{PT} \)-symmetric and symmetric interaction

**Symmetric point interaction at** \( x = \pm l \)

\[
(U - I)\Psi(l) + i L_0 (U + I)\Psi'(l) = 0
\]

\[
\Psi(l) = \begin{pmatrix} \psi(l) \\ \psi(-l) \end{pmatrix}, \quad \Psi'(l) = \begin{pmatrix} \psi'(l) \\ -\psi'(-l) \end{pmatrix}
\]

\( U \) is unitary matrix

**\( \mathcal{PT} \)-symmetric point interaction at** \( x = 0 \)

\[
\psi(0+) = e^{i\phi}\psi(0-)
\]

\[
\psi'(0+) = e^{-i\phi}\psi'(0-)
\]
$\mathcal{PT}$-symmetric and symmetric interaction

**Spectrum**

- discrete ($\lambda = k^2$) if $\phi \neq \pm \pi/2$

$$\cos \phi \left( P_1(U) - 2ikL_0 P_2(U) \cos 2kI + k^2 L_0^2 P_3(U) \sin 2kI \right) +$$

$$+ 2ikL_0 \left( u_{12} + u_{21} + i(u_{11} - u_{22}) \sin \phi \right) = 0$$

- $\phi = \pm \pi/2$
  - empty if $u_{12} + u_{21} \pm i(u_{11} - u_{22}) \neq 0$
  - entire $\mathbb{C}$ if $u_{12} + u_{21} \pm i(u_{11} - u_{22}) = 0$

Dirichlet $U = -I$, Neumann $U = I$, Robin $U = \alpha I$, $\alpha \in \mathbb{R}$
Two $\mathcal{PT}$-symmetric interactions

\[
\begin{align*}
\psi(0^+) &= e^{i\phi_1} \psi(0^-) \\
\psi'(0^+) &= e^{-i\phi_1} \psi'(0^-) \\
\begin{pmatrix}
\psi(l) \\
\psi'(l)
\end{pmatrix} &= B 
\begin{pmatrix}
\psi(-l) \\
\psi'(-l)
\end{pmatrix} \\
B &= \begin{pmatrix}
\sqrt{1 + b_2 c_2 e^{i\phi_2}} & b_2 \\
c_2 & \sqrt{1 + b_2 c_2 e^{-i\phi_2}}
\end{pmatrix}
\end{align*}
\]
Two $\mathcal{PT}$-symmetric interactions

**Spectrum**

- discrete ($\lambda = k^2$) if
  - $\phi_1 \neq \pm \pi/2, \phi_2 \neq \pm \pi/2$
  - $\phi_1 \neq \pm \pi/2, \phi_2 = \pm \pi/2$ and $b_2 \neq 0$ or $c_2 \neq 0$

\[
\cos \phi_1 \left( (b_2 k^2 - c_2) \sin 2kl + 2k \sqrt{1 + b_2 c_2} \cos \phi_2 \cos 2kl \right) + 2k \left( \sqrt{1 + b_2 c_2} \sin \phi_1 \sin \phi_2 - 1 \right) = 0.
\]

- empty if $\phi_1 = \pm \pi/2$ and $\sqrt{1 + b_2 c_2} \sin \phi_2 - 1 \neq 0$
- entire $\mathbb{C}$ if $\phi_1 = \pm \pi/2$ and $\sqrt{1 + b_2 c_2} \sin \phi_2 - 1 = 0$
- $b_2 = c_2 = 0$
  - empty if $\phi_1 = \pm \pi/2$ and $\phi_2 \neq \pm \pi/2$
  - entire $\mathbb{C}$ if $\phi_1 = \phi_2 = \pm \pi/2$
Conclusions

- Spectral properties of $\mathcal{PT}$-symmetric operators can be very rich.
- $\mathcal{PT}$-symmetry, pseudo-Hermiticity, $J$-self-adjointness do not guarantee non-empty spectrum, countable point spectrum, spectral decomposition.
- Examples of $\mathcal{PT}$-symmetric point interactions:
  - line $\mathbb{R} - \sigma = \mathbb{C}$, $\sigma_c = [0, \infty)$, $\sigma_p = \mathbb{C} \setminus [0, \infty)$
  - finite interval $(-l, l) - \sigma = \emptyset$ versus $\sigma = \sigma_p = \mathbb{C}$