Geometry of the Berry Phase

.... a concise $\mu$-seminar exposition ....

Denis Kochan

Comenius University

October 11, Řež
The problem formulation

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- equivalently, there $\exists$ a collection of smooth maps $\{\Phi_m\}$

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\Phi_m : q \mapsto \{|m(q)\rangle, E_m(q)\}, \quad \text{where} \quad \hat{H}(q)|m(q)\rangle = E_m(q)|m(q)\rangle
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$\Phi_m : q \mapsto \{|m(q)\rangle, E_m(q)\}$, where $\hat{H}(q)|m(q)\rangle = E_m(q)|m(q)\rangle$

- Remark I.: (gauge transformation in given eigenvalue sector)

$\Phi_m \mapsto \Phi'_m : q \mapsto \{|\tilde{m}(q)\rangle = e^{i\alpha_m(q)}|m(q)\rangle, E_m(q)\}$
Geometrical overview - fiber bundle perspective

\[ \mathcal{Q} \times \mathcal{H} \]

\( \{ \mathcal{H}, \hat{H}(q) \} \)

\( \{ \mathcal{H}, \hat{H}(q') \} \)

\[ |m(q)\rangle \]

\[ |k(q)\rangle \]

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The problem formulation continues ....

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- a circuit (smooth loop) $C$ in the parameter space $\mathcal{Q}$

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What is the task:
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What is the task:

- to find a solution $|\Psi(q_C(t = T))\rangle$ of the Schrödinger equation:

\[ i\hbar \partial_t |\Psi(t)\rangle = \hat{H}(q_C(t))|\Psi(t)\rangle + \text{ initial condition} \]
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Geometry of the Berry Phase
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- Result in a system of the first-order diff. eq. for $\{B_m(t)\}$:

$$\dot{B}_m = -B_m\langle m|\hat{m}\rangle - \sum_{k \neq m} B_k \frac{\langle m| \frac{d}{dt}\hat{H}(t)|k\rangle}{E_k - E_m} \exp\{-\frac{i}{\hbar} \int_0^t (E_k - E_m)d\tau\}$$
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- Together with set of initial conditions: $\{B_m(t = 0) = \delta_{mn}\}$
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- adiabatic theorem: $|\psi(t)\rangle = e^{-i\gamma_n(t)}|n(q_C(t))\rangle$, where

$$\gamma_n(t) = \frac{1}{\hbar} \int_{0}^{t} E_n(\tau) d\tau + \text{Im} \int_{C(t)} \langle n(q)| \frac{d}{dq} n(q) \rangle dq =: \gamma_n^D + \gamma_n^G$$
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- Remark II.: for almost 50 years was $\gamma_n^G$ ignored (!)
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Born-Fock gauge fixing: $|n(q)\rangle \mapsto |\tilde{n}(q)\rangle = e^{-i\gamma_n^G} |n(q)\rangle$
Few facts ....
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- **M. Berry**: it **fails globally**, if fundamental group $\pi_1(C) \neq \{e\}$

- **some fundamental groups**:
  
  \[
  \begin{align*}
  \pi_1(\text{point}) &= \{e\} & \pi_1(\text{line}) &= \{e\} \\
  \pi_1(\text{circle}) &= \mathbb{Z} & \pi_1(\text{**}) &= \mathbb{Z} \ast \mathbb{Z} / \{a^3 \ast b^{-2}\}
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$$|\psi(0)\rangle = a_n |n\rangle + a_m |m\rangle \leadsto |\psi(T)\rangle = a_n e^{-i \gamma n} |n\rangle + a_m e^{-i \gamma m} |m\rangle$$
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  measurement of $\langle \Psi(T)|\hat{A}|\Psi(T)\rangle$ for some observable $\hat{A}$

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contributes only if $[\hat{A}, \hat{H}(T)] \neq 0$
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- $n$-th spectral bundle (subset of the trivial vector bundle $\mathcal{Q} \times \mathcal{H}$)

$$\mathcal{L}_n := \{(q, |\Psi\rangle) \in \mathcal{Q} \times \mathcal{H}, \langle \psi | \psi \rangle = 1 \text{ and } \hat{H}(q)|\psi\rangle = E_n(q)|\psi\rangle\}$$
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- RLC connection uses ambient space parallelism and metric:
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- picture of the Berry-Barry parallel transport:
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- Picture of the Berry-Barry parallel transport:
### Adiabaticity & (An)holonomy

#### Gauge potential $A_\mu$

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Gauge potential $A_\mu$

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- what is a parallel “state-mate” of $|n(q)\rangle$ after an infinitesimal parameter translation $q \mapsto q' = q + \delta v$?
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\delta \varphi = A_\mu(q) \cdot \delta v^\mu = i \left\langle n(q) \left| \frac{\partial}{\partial q^\mu} n(q) \right\rangle \right. \cdot \delta v^\mu
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- Remark III.: $\langle n(q)|\partial_{q^\mu} n(q)\rangle$ is pure imaginary quantity, thus

$$A = A_\mu(q) dq^\mu = -\text{Im} \langle n(q)|\partial_{q^\mu} n(q)\rangle \ dx^\mu$$
More realistic perspective

\[ U(1) \simeq S^1 \]

|n(q)\rangle \quad \delta v \quad |n(q')\rangle \parallel_{BB} \quad \delta \varphi \quad |n(q')\rangle \quad \Phi_n \quad \delta v \quad q \quad q' = q + \delta v \quad Q \quad \mathcal{L}_{n}\]
Berry phase as (an)holonomy of $A_\mu$
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$$A_\mu \mapsto A_\mu - \partial_\mu \alpha_n \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \mapsto F_{\mu\nu}$$
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- phase acquired over a circuit $C$ is the (an)holonomy of $A_\mu$, i.e.

  $$\gamma_n^G = \oint_C A_\mu dq^\mu = \int_\Sigma \frac{1}{2} F_{\mu\nu} dq^\mu \wedge dq^\nu , \quad \text{where} \quad \partial \Sigma = C$$
Berry phase as (an)holonomy of $A_\mu$

- **gauge transformation**: \( (q, |n(q)\rangle) \mapsto \left( q, e^{i\alpha_n(q)}|n(q)\rangle \right) \)
  
  \[ A_\mu \mapsto A_\mu - \partial_\mu \alpha_n \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \mapsto F_{\mu\nu} \]

- if the curvature (electromagnetic stress tensor) \( F_{\mu\nu} \neq 0 \), then BB transport is path dependent

- phase acquired over a circuit $C$ is the (an)holonomy of $A_\mu$, ie.

  \[ \gamma^G_n = \oint_C A_\mu dq^\mu = \int_{\Sigma} \frac{1}{2} F_{\mu\nu} dq^\mu \wedge dq^\nu \quad , \quad \text{where} \quad \partial \Sigma = C \]

  magnetic flux through $\Sigma$
Conclusion
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in classical mechanics, there is as well in adiabatic regime an analog of the Berry phase called Hannay phase (or angel)
a čo povedať fakt úplne na záver?
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**thanks for your attention!**