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Structure and separability of driven and triangular systems of Newton equations

Abstract

The classical separability theory of potential Newton equations $\ddot{q} = -\nabla V(q)$ and of the related natural Hamiltonians $H = \frac{1}{2} p^2 + V(q)$ has been a cornerstone of almost all exactly solved problems in Analytical Mechanics and a pivotal factor in building early theory of quantisation in Quantum Mechanics. This theory is well summarised in recent papers by Benenti, Chanu, Rastelli in JMP (2002, 2003).

A natural generalisation of this theory are (discovered in Linköping 1999) systems of quasipotential Newton equations of the form $\ddot{q} = M(q) = -A(q)^{-1}\nabla k(q)$, $q \in R^n$, $A(q) -$ Killing matrix. If $\ddot{q} = M(q)$ admit two quadratic integrals of motion then there are n quadratic integrals of motion and the equations are completely integrable. These Newton equations are then characterised through a certain Poisson pencil and, equivalently, through a system of $\frac{1}{2} n(n-1)$ 2nd order PDE´s - the Fundamental Equations, which for potential forces reduce to the well-known Bertrand-Darboux equations. We have also shown that bi-quasipotential Newton equations are separable in new types of coordinates given by nonconfocal quadric surfaces.

The theory of bi-quasipotential Newton equations have been soon generalised by Sarlet and Crampin (2001) to the framework of Riemannian manifolds as geodesic equations with a forcing term. In 2005 S.Benenti discovered that the bi-quasipotential property of Newton equations leads to the Levi-Civita dynamically equivalent systems on Riemannian manifolds.

I shall review main theorems of theory of quasipotential Newton equations and will talk about an interesting subclass of driven Newton equations $\ddot{y} = M_{\uparrow}(y)$, $\ddot{x} = M_{\downarrow}(y,x) = -\nabla_x V(y,x)$ for which knowledge of a single quadratic integral $E = \dot{q}^t cofG\dot{q} + k(q)$, $q = (y,x) \in R^n$ is sufficient for separability of the time dependent Hamilton-Jacobi equation corresponding to Newton equations of the form $\ddot{x} = -\nabla_x V(y(t),x)$.

For the subclass of triangular systems of Newton equations $\ddot{q}_k = M_k(q_1,...,q_k)$, $k = 1,...,n$ even a stronger $(1 \Rightarrow n)$ theorem is valid. It says that knowledge of one quadratic integral implies existence of n quadratic integrals and the system is solvable by separation of variables. The emerging separation coordinates are described for $n = 2$ and for $n = 3$.